# On Semiclassical (Zero Dispersion Limit) Solutions of the Focusing Nonlinear Schrödinger Equation


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**Abstract**

We calculate the leading-order term of the solution of the focusing nonlinear (cubic) Schrödinger equation (NLS) in the semiclassical limit for a certain one-parameter family of initial conditions. This family contains both solitons and pure radiation. In the pure radiation case, our result is valid for all times $t \geq 0$. We utilize the Riemann-Hilbert problem formulation of the inverse scattering problem to obtain the leading-order term of the solution. Error estimates are provided. © 2004 Wiley Periodicals, Inc.


## 1 Introduction

We study the semiclassical limit of the focusing nonlinear Schrödinger equation (NLS)

$$i\varepsilon q_t + \left(\frac{\varepsilon^2}{2}\right) q_{xx} + |q|^2 q = 0\,, \tag{1.1}$$

subject to a one-parameter family of initial conditions

$$q(x, 0, \varepsilon) = A(x) e^{iS(x)/\varepsilon} \tag{1.2}$$

with $A(x) = -\operatorname{sech} x$, $S' = -\mu \tanh x$, $S(0) = 0$, where $\mu \geq 0$. The initial value problem for the cubic NLS equation with decaying initial data was solved in [40] through the introduction of an appropriate Lax pair. The semiclassical problem (i.e., small $\varepsilon$) has been the object of research in the last 20 years, the traditional insight being that the modulational instability [18] would lead to a seemingly chaotic type of behavior. Recent numerical studies [4, 29], however, have indicated that a great amount of order may persist as the system evolves.

The first analytic breakthrough [24], using initial data (1.2) with $\mu = 0$ for which the scattering data were known [31], provided insights into the observed structures including formulae for the short-time evolution of the initial data and for


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the passage through a first break at some point $(x, t)$ assuming that such a break exists. One of the assumptions is that $\varepsilon$ takes a sequence of values for which the problem has only pure soliton solutions. The singular passage from the pure soliton scattering problem to a limiting continuum Riemann-Hilbert problem (RHP) with contour coinciding with the arc of the soliton poles was established rigorously in [28].

In the present study we prove the existence of observed waveforms and provide the leading behavior of the solution when $\mu > 0$, in both the pure radiation case (globally in time) and the radiation/soliton case (up to the second break). We believe that our results extend naturally to the case $\mu = 0$. The scattering data of our one-parameter family of initial data (1.2) was derived explicitly in [33], in which the Zakharov-Shabat eigenvalue problem is solved explicitly in terms of hypergeometric functions.

Our analysis utilizes the “steepest descent" approach [15, 16] and its essential extension [14] (see also [13]), which introduced a systematic mechanism for deriving the $g$-function (introduced in [16]) allowing the treatment of fully nonlinear waveforms. The method has been applied in a variety of problems; see [8, 9, 10, 11]. We treat the case of the one-parameter family of initial data with a procedure that is appropriate for the study of a wide family of scattering data with certain global analytic properties.

Compared to the initial value problem for the small dispersion Korteweg–de Vries (KdV) equation,

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0\,, \quad u(x,0) = u_0(x)\,, \quad \varepsilon \to 0\,, \tag{1.3}$$

our initial value problem for the semiclassical focusing NLS equation displays great similarity. We prove that fully nonlinear oscillations in the small ($O(\varepsilon)$) spatial and temporal scale emerge. As in KdV, they consist of modulated waves described in terms of theta functions with the number of wave phases independent of $\varepsilon$. The essential difference from KdV is the modulational instability [18] of these NLS waves. As a result of the modulational instability, the absence of some global analyticity property in the scattering data is expected to break the order in the wave structures of the solution.

The weak small-dispersion KdV limit was first calculated in the pure soliton case [25], where the number of solitons $N$ is equal to the number of eigenvalues of the corresponding Schrödinger operator and $N \sim O(1/\varepsilon)$, $\varepsilon \to 0$. The solution of KdV is given in terms of the $N \times N$ Kay-Moses determinant $\det(I + K)$, a simpler form of the Dyson determinant [5] that applies to pure soliton solutions. It expands [25] into the Fredholm sum $\sum_S \det K_S$ of positive determinants with $S$ ranging over all subsets of the set $\{1, 2, \ldots, N\}$ of eigenvalue indices, where $K_S$ is the determinant that arises from $K$ when all rows and columns indexed by an element of $S$ are discarded. It is shown in [25] that the largest term of the sum produces the weak limit as $\varepsilon \to 0$; the maximizing subset $S^*$ is calculated in its continuum limit through a variational principle. More precisely, since each index corresponds to an eigenvalue of the associated Schrödinger operator, the limit of the maximizer is calculated as a semiclassical density of states constrained not to exceed the full semiclassical density of states of the initial data.

Subsequent to [25], the weak limit for the pure radiation, small-dispersion KdV case was calculated in [36]. The Dyson determinant has a different type of kernel in this case, and the problem is not directly reducible to the [25] case. The periodic case was solved in [37]. Two more systems have been calculated in the small-dispersion limit following the general approach of [25]: the defocusing nonlinear Schrödinger equation in [23] and the Toda lattice in [12].

A similar type of calculation is also used in [7], where the analyticity of a certain potential (this would be the scattering data in the language of our present problem) is shown to lead to a finite number of intervals forming the support of the maximizing density. The first rigorous construction of the support was obtained for KdV in [32] (see also [22] for the latest result) through the analysis of the associated Whitham system. Since our approach is different, we will not go into the large literature on this subject.

In all these studies the limit is weak; in other words, the nonlinear oscillations in the solution are averaged out. The Lax-Levermore procedure was strengthened in [38] (see also [35]) to provide the strong small-dispersion limit, or more precisely, the leading asymptotic behavior of the solution of the above KdV initial value problem as $\varepsilon \to 0$. The result was obtained by complementing the formulation in [25] with a *quantum condition* that forces the number of eigenvalues over each connected component of the support of maximizing density of states to be an integer. The Lax-Levermore maximizing density $\psi^*$ (the equilibrium measure is $\psi^*(z)dz$) then requires the higher-order correction $\psi^* + \varepsilon\overline{\psi}$ with the condition $\int \psi^* + \varepsilon\overline{\psi} =$ integer $\times\ \varepsilon$ over each connected component of the support of $\psi^*$. The variational problem of [25] with the quantum condition has multiple solutions $\overline{\psi} = \overline{\psi}_\alpha$, labeled by the multi-integer $\alpha$ (an integer corresponding to each component of the support). Each $\overline{\psi}_\alpha$ contributes to the limiting determinant. Putting together these contributions, we obtain directly the standard expansion of the theta function that describes the asymptotic waveform. This procedure was carried out for the Toda shock problem in [39].

In all these studies, the asymptotic analysis was carried out on the Gelfand-Levitan-Marchenko-Dyson procedure for solving the inverse scattering problem [5, 20, 26, 27]. On the other hand, the steepest-descent method, used here, applies to the formulation of inverse scattering as a Riemann-Hilbert problem (RHP), an approach initiated in [30]. A matrix $m(z)$ constructed of solutions of the associated linear Zakharov-Shabat eigenvalue system (the first operator of the Lax pair) depends analytically on the spectral variable $z$ at all points of the closed complex plane except on an oriented contour on which it experiences a jump, $m_+ = m_- V$. The $2 \times 2$ jump matrix $V(z)$ (in both KdV and NLS cases) is expressed in terms of the scattering data, and its time evolution is very simple and explicit. The RHP consists in deriving $m(z)$ from $V(z)$.

The linear eigenvalue problem corresponding to the integration of NLS

$$i\varepsilon W' = \begin{pmatrix} z & q \\ \bar{q} & -z \end{pmatrix} W \,, \tag{1.4}$$

where $q = q(x, 0, \varepsilon)$ is referred to as a potential and $z \in \mathbb{C}$ is a spectral parameter, was studied in [33]. In this paper the scattering coefficients $a$ and $b$ (see [40]) and the reflection coefficient $r^{(0)}(z) = b(z)/a(z)$, corresponding to (1.4), were found as products of gamma functions:

$$\begin{aligned} a(z) &= \frac{\Gamma(w)\Gamma(w - w_+ - w_-)}{\Gamma(w - w_+)\Gamma(w - w_-)} \,, \\ b(z) &= -i\varepsilon 2^{-\frac{i\mu}{\varepsilon}} \frac{\Gamma(w)\Gamma(1 - w + w_+ + w_-)}{\Gamma(w_+)\Gamma(w_-)} \,, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} r^{(0)}(z) &= \frac{b(z)}{a(z)} \\ &= -i\varepsilon 2^{-\frac{i\mu}{\varepsilon}} \frac{\Gamma(1 - w + w_+ + w_-)\Gamma(w - w_+)\Gamma(w - w_-)}{\Gamma(w_+)\Gamma(w_-)\Gamma(w - w_+ - w_-)} \,, \end{aligned} \tag{1.6}$$

where

$$w_+ = -\frac{i}{\varepsilon}\Big(T + \frac{\mu}{2}\Big), \quad w_- = \frac{i}{\varepsilon}\Big(T - \frac{\mu}{2}\Big), \quad w = -z\frac{i}{\varepsilon} - \mu\frac{i}{2\varepsilon} + \frac{1}{2} \,, \tag{1.7}$$

and

$$T = \sqrt{\frac{\mu^2}{4} - 1} \,.$$

In the theory of inverse scattering, the coefficient $a(z)$ is defined in the upper $z$ half-plane while $b(z)$ and the reflection coefficient are defined on the real $z$-axis. In the case $0 \le \mu < 2$ the eigenvalue problem (1.4) contains points of discrete spectrum (zeros of $a(z)$) at $z_k = T - i\varepsilon(k - \frac{1}{2})$ with the corresponding norming constants

$$c_k^{(0)} = \frac{b(z_k)}{a'(z_k)} = \operatorname{Res}_{z=z_k} r^{(0)}(z) \,. \tag{1.8}$$

Here $k \in \mathbb{N}$ and $k < \frac{1}{2} + \frac{|T|}{\varepsilon}$. Because of the Schwarz reflection symmetry of the problem, it is sufficient to specify the discrete spectrum in the upper half-plane only.

The *main purpose* of this study is to calculate the leading-order term $q_0(x, t, \varepsilon)$ (with respect to $\varepsilon$) of $q(x, t, \varepsilon)$. The time evolution of the scattering data [40] is very simple and explicit (see below); thus, the calculation of the evolution of the initial value problem (1.1)–(1.2) essentially consists of solving the inverse scattering problem (ISP), i.e., of reconstructing the potential $q = q(x, t, \varepsilon)$ in (1.4) from the explicitly available scattering data at the time $t$. In this paper the ISP is formulated as a (multiplicative) matrix Riemann-Hilbert problem (RHP) on the complex plane of the spectral variable $z$. We approximate the RHP with some model RHP that has an explicit solution $q_0(x,t,\varepsilon)$. We then show that this $q_0(x,t,\varepsilon)$ is a leading asymptotic expression for the solution of the problem in which our initial scattering data are replaced by its Stirling approximation. The tools developed in this paper are in many cases sufficient for the calculation of the higher-order terms; however, we have not included such calculations.

For any given pair $(x,t)$, our procedure reduces the construction of $q_0(x,t,\varepsilon)$ to the solution of the model RHP on a contour that consists of $2N+1$ arcs $\{\gamma_{m,j}\}_{j=-N}^{N}$ (we refer to them as "main arcs") interlaced with $2N$ "complementary arcs" $\{\gamma_{c,j}\}$, $j=\pm 1,\pm 2,\dots,\pm N$ (see Figure 2.3). The arcs, as well as their endpoints $\alpha_j = a_j + ib_j$, $j=0,1,\dots,4N+1$, depend on $x$ and $t$ but not on $\varepsilon$. On each of these arcs, whose determination is an important part of our procedure, the $2\times 2$ jump matrix of the model RHP is constant with respect to $z$, but depends on the parameters $x$, $t$, and $\varepsilon$. A major ingredient of the solution to the model RHP is the radical $R(z;x,t)=\prod_{j=0}^{4N+1}\sqrt{(z-\alpha_j)}$ with branch cuts along the main arcs, as well as the associated two-sheeted Riemann surface $\mathcal{R}(x,t)$. The solution of the model RHP is obtained explicitly through the dual basis of holomorphic differentials $\omega$ of $\mathcal{R}(x,t)$ and the corresponding Riemann theta function $\theta(u)$; see Section 7. In a sense, we study the evolution of $q(x,t,\varepsilon)$ through the evolution of $\mathcal{R}(x,t)$, which identifies $q_0(x,t,\varepsilon)$ in a neighborhood of $(x,t)$ as an $N$-phase NLS solution. The genus $2N$, $N=0,1,\dots$, of $\mathcal{R}(x,t)$ is physically important as it specifies the number of oscillatory phases of the solution. By a mild abuse of terminology we call it "the genus of the solution $q_0(x,t,\varepsilon)$" or simply "the genus." A line on the $(x,t)$–plane separating regions of different genus is called a "breaking curve."

The main result of the paper, stated below, refers to the case $\mu>0$, $x\geq 0$, $t\geq 0$. We have realized our above procedure in a rigorous sense in the cases of genus 0 and 2 (due to a reflection symmetry with respect to the real axis, the genus must be even and $\overline{\gamma}_{m,j}=\gamma_{m,-j}$, $\overline{\gamma}_{c,j}=\gamma_{c,-j}$, $j=1,2,\dots,N$, $\overline{\gamma}_{m,0}=\gamma_{m,0}$). It follows from (1.1)–(1.2) that the solution $q(x,t,\varepsilon)$ is an even function in $x$ for all $t$. Our main theorem provides exact expressions for $q_0(x,t,\varepsilon)$ through the branch points $\alpha_j(x,t)$, the Riemann theta function $\theta$, and holomorphic differentials $\omega$, associated with the basic cycles of the Riemann surface $\mathcal{R}(x,t)$, and real quantities $W_i(x,t)$, $\Omega_i(x,t)$, $i=1,2,\dots,N$. In the genus 0 case, an explicit expression for $\alpha_0=a+ib$ is provided by (4.27)–(4.28), whereas in general the branch points $\alpha_j(x,t)$ are determined through the system of moment (3.5) and integral conditions (3.9), and the quantities $W_i=W_i(x,t)$ and $\Omega_i=\Omega_i(x,t)$ are determined by (3.8). The expression for $q_0(x,t,\varepsilon)$ becomes somewhat simpler if we consider the theta function $\theta$ and holomorphic differentials $\omega$, associated with the basic cycles of the Riemann surface $\tilde{\mathcal{R}}(x,t)$ (Figure 8.2) that consists of $2N+1$ vertical cuts $\tilde{\nu}_j$ connecting the corresponding (complex-conjugated) endpoints of the main arcs $\gamma_{m,j}$ and $\overline{\gamma}_{m,j}$, $j=0,1,\dots,N$; see Section 8. In the expression for the genus $2N$ solution (Section 8; also the main theorem with $N=1$) the notation $W$ stands for the $\mathbb{R}^{2N}$ vector $(W_{-N},\dots,W_{-1},W_1,\dots,W_N)^{\mathsf{T}}$ with $W_{-i}=W_i$, as well as for the

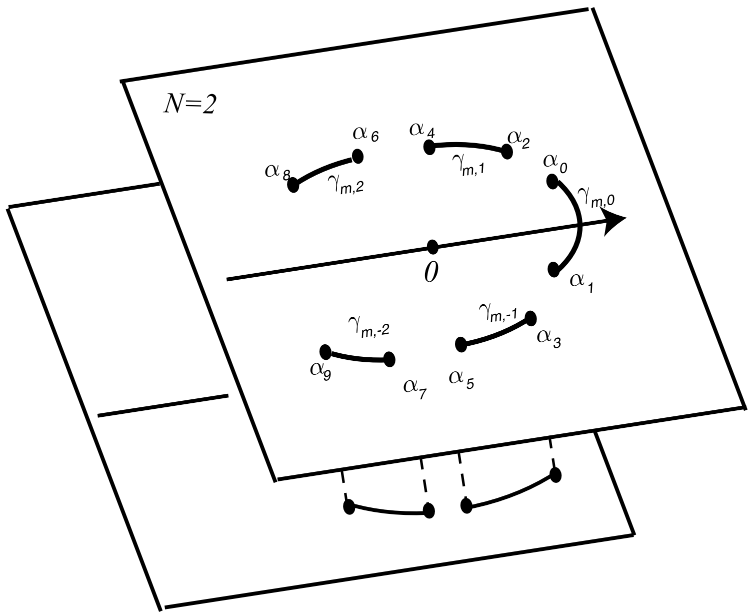


FIGURE 1.1. Riemann surface $\mathcal{R}(x,t)$.

piecewise constant scalar function on $\gamma_m = \bigcup_i \gamma_{m,i}$ that takes the value $W_i$ on the arc $\gamma_{m,i}$ and the value $W_0 = 0$ on the arc $\gamma_{m,0}$. Similarly, $\Omega$ denotes the $\mathbb{R}^{2N}$ vector $(\Omega_{-N}, \dots, \Omega_{-1}, \Omega_1, \dots, \Omega_N)$ with $\Omega_{-i} = \Omega_i$, as well as the piecewise constant function that takes the value $\Omega_i$ on the arc $\gamma_{c,i}$.

THEOREM 1.1 (Main Theorem) *There exists a breaking curve $t = t_0(x)$, $x \in \mathbb{R}$, with the following properties*:

(i) *The genus of the solution $q_0(x,t,\varepsilon)$ is zero below the curve, i.e., in the region $0 \le t < t_0(x)$.*

(ii) *In the solitonless* (*pure radiation*) *case $\mu \ge 2$, the solution in the entire region above the breaking curve has genus exactly* 2. *In the case that includes solitons, i.e., $\mu < 2$, there exists some function $t_1(x)$, $x \in \mathbb{R}$, $t_0(x) < t_1(x) \le \infty$, such that the genus equals* 2 *in the region $t_0(x) < t < t_1(x)$*; *see Figure* 1.2.

(iii) *The breaking curve is an even function, smooth and monotonically increasing for $x > 0$ with the asymptotic behavior $t_0(x) \sim \frac{x}{2\mu}$ as $x \to +\infty$ and $t_0(x) = \frac{1}{2(\mu+2)} + 2\sqrt{\mu+2}\tan\frac{\pi}{5}x + o(x)$ as $x \to 0^+$.*

(iv) *In the genus* 0 *region* $(0 \le t < t_0(x))$,

$$q_0(x,t,\varepsilon) = \Im\alpha_0(x,t)e^{-2\frac{i}{\varepsilon}\int_0^x \Re\alpha_0(s,t)ds}. \tag{1.9}$$

(v) *In the genus* 2 (*i.e.,* $N = 1$) *region* ($t_0(x) < t < t_1(x)$),

$$q_0(x,t,\varepsilon) = \frac{2\theta(0)\theta(d_1)\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi}+d_1)\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi})}{\theta(u(\infty))\theta(u(\infty)+d_1)\theta(\frac{\hat{\Omega}}{2\pi}+d_1)\theta(\frac{\hat{\Omega}}{2\pi})} \times \left[\sum_{j=0}^{2}(-1)^j \Im\alpha_{2j} - i\nabla\ln\frac{\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi}+d_1)\theta(u(\infty))}{\theta(u(\infty)+d_1)\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi})}\cdot\omega^0\right] \times e^{\frac{2}{\varepsilon\pi}\left(\int_{\gamma_m}\frac{f_0^{(0)}(\zeta)+\frac{\pi}{2}\varepsilon+x\zeta+2t\zeta^2+W}{R_+(\zeta)}\zeta^2\,d\zeta+\int_{\gamma_c}\frac{\Omega}{R(\zeta)}\zeta^2\,d\zeta+i\pi\Omega\right)}, \tag{1.10}$$

*where $\gamma_m$ and $\gamma_c$ denote the union of all main and all complementary arcs, respectively*; *the theta functions and the basic holomorphic differentials $\omega$, dual to $\boldsymbol{A}$-cycles, are associated with the hyperelliptic Riemann surface $\tilde{\mathcal{R}}(x,t)$* (*see Figures* 8.2 *and* 8.3), *and the vector $\omega^0 \in \mathbb{C}^2$ is the leading coefficient of $\omega$*; $u(z) = \int_{\alpha_1}^{z}\omega$; $\hat{\Omega}_1 = -\frac{2}{\varepsilon}W$, $\hat{\Omega}_2 = -\frac{2}{\varepsilon}(W+\Omega)$, $d_1 = \frac{1}{2}(1,1)^{\mathsf{T}}$; $f_0^{(0)}(z)$ *is the leading-order term of* $\frac{i}{2\varepsilon}\ln r^{(0)}(z)$ *as* $\varepsilon \to 0$; *and* $R(z) = \prod_{j=0}^{5}\sqrt{(z-\alpha_j)}$, *and the branch* $R_+(z) \to -z^3$ *as* $z \to \infty$. *The real constants $W$ and $\Omega$ are determined through* (3.8), *where* $W = W_1$ *and* $\Omega = \Omega_1$. *The value $W_0$ of $W$ on $\gamma_{m,0}$ is* 0.

*If, additionally, we assume that the function* $\lambda(z) - \lambda^{-1}(z)$, *where $\lambda(z)$ is expressed through the endpoints of the vertical cuts $\tilde{\nu}_j$ of $\tilde{\mathcal{R}}(x,t)$ as*

$$\lambda(z) = \left[\frac{(z-\alpha_0)(z-\alpha_3)(z-\alpha_4)}{(z-\alpha_1)(z-\alpha_2)(z-\alpha_5)}\right]^{\frac{1}{4}},$$

*has two distinct zeros $z_1$ and $z_2$, then the expression for $q_0(x,t,\varepsilon)$ can be written as*

$$q_0(x,t,\varepsilon) = \frac{\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi}-d)\theta(u(\infty)+d)}{\theta(u(\infty)-\frac{\hat{\Omega}}{2\pi}+d)\theta(u(\infty)-d)} \times e^{\frac{2}{\varepsilon\pi}\left(\int_{\gamma_m}\frac{f_0^{(0)}(\zeta)+\frac{\pi}{2}\varepsilon+x\zeta+2t\zeta^2+W}{R_+(\zeta)}\zeta^2\,d\zeta+\int_{\gamma_c}\frac{\Omega}{R(\zeta)}\zeta^2\,d\zeta+i\pi\Omega\right)}\sum_{j=0}^{2}(-1)^j\Im\alpha_j\,. \tag{1.11}$$

*Here*

$$d = -\int_{\alpha_2}^{X_2(z_1)}\omega_1 - \int_{\alpha_5}^{X_2(z_2)}\omega_2\,,$$

*where $X_2(z)$ is the preimage of $z$ on the second sheet of the hyperelliptic surface $\tilde{\mathcal{R}}(x,t)$. Similar to* (1.10), *an expression for $q_0(x,t,\varepsilon)$ through theta functions, associated with the Riemann surface $\mathcal{R}(x,t)$, can be found in* (1.13) *below. The formulae* (1.10), (1.11), *and* (1.13) *can be extended to genus $2N$ regions in the $(x,t)$–plane*; *see Theorem* 8.1, *Theorem* 8.2, *and Theorem* 7.1, *respectively.*

(vi) *The accuracy of the above leading-term approximations is given by*

$$|q(x,t,\varepsilon) - q_0(x,t,\varepsilon)| = O(\varepsilon)\,, \tag{1.12}$$

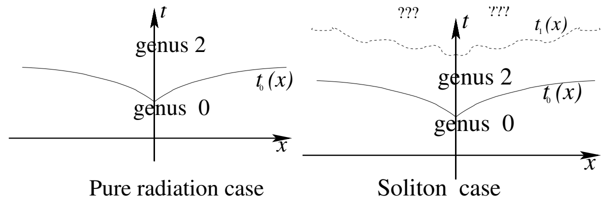


FIGURE 1.2. Different genus regions in the $(x, t)$–plane; linear behavior as $x \to 0^+$, $x \to \infty$.

*locally uniformly in x and t away from the breaking curve.*

*Comments.*

- The theorem is proven for the initial potential, which slightly differs from (1.2). Namely, we replace the initial reflection coefficient
$$r^{(0)}(z) = \exp\left(-\frac{2i}{\varepsilon} f^{(0)}(z)\right)$$
for the ISP by
$$r_0^{(0)}(z) = \exp\left(-\frac{2i}{\varepsilon}(f_0^{(0)}(z) + \frac{\pi}{2}\varepsilon)\right) \quad \text{on } z \in \mathbb{R},$$
where $f^{(0)}(z)$ has asymptotic expansion in $\varepsilon$ with the two leading terms $f_0^{(0)}(z) + \frac{\pi}{2}\varepsilon$. Corollary 4.8 below shows that the initial potential, corresponding to $r_0^{(0)}(z)$, coincides with (1.2) up to $O(\varepsilon)$. However, the general question of the stability of solutions to (1.1) with respect to small variations of initial data is a delicate problem that will not be considered in the paper.
- In the case $\mu < 2$, regions of genus greater than 2 above the breaking curve may exist. The study of this case remains in the focus of our ongoing research.
- Another expression for $q_0(x, t, \varepsilon)$ in the genus 2 region is given by

$$\begin{aligned} q_0(x,t,\varepsilon) = {} & \frac{\theta(0)\theta(d_1)\theta(u(\infty) + \frac{\hat{W}}{2\pi} + d_1)\theta(u(\infty) + \frac{\widehat{W}}{2\pi})}{\theta(u(\infty))\theta(u(\infty) + d_1)\theta(\frac{\widehat{W}}{2\pi} + d_1)\theta(\frac{\widehat{W}}{2\pi})} \\ & \times \Bigg[ \Im \sum_{j=1}^{2} (\alpha_{4j} - \alpha_{4j-2}) \\ & \qquad - i\nabla \ln \frac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d_1)\theta(u(\infty))}{\theta(u(\infty) + d_1)\theta(u(\infty) + \frac{\widehat{W}}{2\pi})} \cdot \omega^0 \Bigg] \\ & \times e^{\frac{2}{\varepsilon\pi}\int_{\gamma_m} \frac{f_0^{(0)}(\zeta) + \frac{\pi}{2}\varepsilon + x\zeta + 2t\zeta^2}{R_+(\zeta)} P(\zeta)d\zeta}, \end{aligned} \tag{1.13}$$

where $\gamma_m$ and $\gamma_c$ denote the union of all main and all complementary arcs, respectively; theta functions and the basic holomorphic differentials $\omega$, dual to $\boldsymbol{\alpha}$-cycles $\boldsymbol{\alpha}$, are associated with the hyperelliptic Riemann surface $\mathcal{R}(x,t)$, and the vector $\omega^0 \in \mathbb{C}^2$ is the leading coefficient of $\omega$; $u(z) = \int_{\alpha_1}^{z} \omega$; $\widehat{W} = -2\frac{i}{\varepsilon}(W + 2\int_{\gamma_c} \Omega\omega)$; $d_1 = \frac{1}{2}(1,1)^{\mathsf{T}}$; $f_0^{(0)}(z)$ is the leading-order term of $\frac{i}{2\varepsilon}\ln r^{(0)}(z)$ as $\varepsilon \to 0$; the quadratic polynomial $P(z) = z^2 + \sum_{j=\pm 1} \int_{\gamma_{m,j}} \frac{\zeta^2\, d\zeta}{R_+(\zeta)} \frac{\omega_j}{dz}$, where $R(z) = \prod_{j=0}^{5} \sqrt{(z-\alpha_j)}$ and the branch $R_+(z) \to -z^3$ as $z \to \infty$. The real constants $W$ and $\Omega$ are determined through (3.8).

The study is divided into the following sections: an outline of the procedure, the construction of functions $g$ and $h$, prebreak evolution, the breaking curve, the higher-genus region, the model RHP, and accuracy estimates. Some calculations and background information can be found in the appendix.

# 2 Outline of the Procedure

## 2.1 The Riemann-Hilbert Problem $P^{(0)}$ for Focusing NLS

The inverse scattering procedure solves NLS via a multiplicative Riemann-Hilbert problem (RHP). Generally, such an RHP is set as follows: given an oriented contour $\Gamma \subset \mathbb{C}$ and a square matrix function $V(z)$ defined on $\Gamma$ and satisfying $\|V^{\pm}\|_{L^\infty(\Gamma)} <$ const, find a matrix function $m(z)$ such that

(1) $m$ is analytic and invertible in $\mathbb{C} \setminus \Gamma$ with continuous boundary values $m_\pm(z)$, $z \in \Gamma$, obtained by approaching $\Gamma$ from the positive and negative side, respectively;
(2) $m$ approaches the identity matrix $I$ as $z \to \infty$, and
(3) $m$ satisfies the jump condition $m_+ = m_- V$ on $\Gamma$.

The matrix $V(z)$ is called the jump matrix. For more details about the RHP, refer to [6, 41].

We label our RHP $P^{(0)}$, the solution matrix $m^{(0)}$, and the jump matrix $V^{(0)}$:

$$\text{(2.1)} \quad \text{RHP } P^{(0)} : m_+^{(0)}(z) = m_-^{(0)} V^{(0)}(z)\,, \quad z \in \Sigma^{(0)}\,, \quad m^{(0)} \to I \ \text{ as } z \to \infty\,.$$

Our contour $\Gamma = \Sigma^{(0)}$ is the union of $\mathbb{R}$ and small circles $C_k$, $k = \pm 1, \pm 2, \dots, \pm n$, encircling each point of discrete spectrum $z_k$. Negative indices correspond to eigenvalues in the lower half-plane. The real axis has the natural orientation, and the circles $C_k$ have positive (counterclockwise) and negative orientation for positive and negative $k$, respectively; see Figure 2.1. The jump matrix $V^{(0)}$, which depends on the parameters $x$, $t$, and $\varepsilon$, is given by

$$\text{(2.2)} \quad V^{(0)} = \begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix} \quad \text{for } z \text{ real}\,, \quad V^{(0)} = \begin{pmatrix} 1 & 0 \\ \frac{c_k}{z-z_k} & 1 \end{pmatrix} \quad \text{for } z \in C_k\,,$$

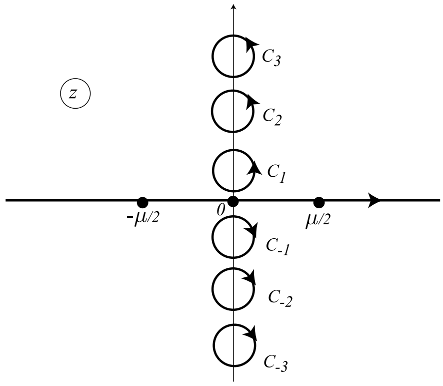


FIGURE 2.1. Contour $\Sigma^{(0)}$.

where

$$r = r(z, x, t) = r^{(0)}(z) e^{(2ixz + 4itz^2)/\varepsilon}, \quad c_k = c_k^{(0)} e^{(2ixz_k + 4itz_k^2)/\varepsilon}. \tag{2.3}$$

Formulae (2.2) and (2.3) reflect the known time evolution of the spectral data; see [40]. The evolution of the potential $q(x, 0, \varepsilon)$, i.e., the solution to the initial value problem (1.1)–(1.2), is given by

$$q(x, t, \varepsilon) = -2(m_1^{(0)})_{1,2} = -2 \lim_{z \to \infty} z(m^{(0)}(z) - I)_{1,2}, \tag{2.4}$$

where $(m)_{ij}$ denotes the $ij^{\text{th}}$ entry of the matrix $m$ and $m^{(0)} = I + m_1^{(0)}/z + O(z^{-2})$; see [6, 41]. We obtain the results of our main theorem by performing a chain of transformations of the RHP whose effect is to peel off the leading contribution to the solution of NLS leading to the problem $P^{(\text{err})}$ ("err" stands for error) whose jump matrix converges uniformly to the identity as $\varepsilon \to 0$ and whose contribution to the solution we estimate. We label the transformations $P^{(0)} \to P^{(1)} \to P^{(2)} \to P^{(3)} \to P^{(4)} \to P^{(\text{err})}$.

(1) $P^{(0)} \to P^{(1)}$ is a factorization of the jump matrix and concomitant contour deformation.
(2) $P^{(1)} \to P^{(2)}$ replaces the scattering data with its Stirling approximation and thus redefines the initial value problem. This study solves the redefined problem. We show that the initial function $q(x, 0, \varepsilon)$ corresponding to the redefined data equals the original initial data to leading order.
(3) $P^{(2)} \to P^{(3)}$ introduces a phase function $g$; the requirement that the jump matrix has certain factorization and decay properties determines $g$.
(4) $P^{(3)} \to P^{(4)}$ consists of contour deformations that utilize the above factorization and decay properties; the contour of $P^{(4)}$ is the union of two subcontours; on the first subcontour the jump matrix is piecewise constant and on the second it converges to the identity as $\varepsilon \to 0$; the convergence is uniform outside of the neighborhood of a number of points.

(5) $P^{(4)} \to P^{(\mathrm{err})}$ peels off the contribution from the first subcontour. $P^{(\mathrm{err})}$ has a jump matrix that equals $I + O(\varepsilon)$. We obtain a rigorous estimate for its solution and its contribution to the solution of NLS.

## 2.2 Initial Jump Matrix Factorization and Contour Deformation: The Riemann-Hilbert Problem $P^{(1)}$

We factor the jump matrix $V^{(0)}$

$$V^{(0)} = \begin{pmatrix} 1 & \bar{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \tag{2.5}$$

for $z < \frac{\mu}{2}$ on the real axis. The point $z = \frac{\mu}{2}$ is special. There is an infinite sequence of nonsoliton poles of $r(z)$ (they are poles of $b(z)$) situated on the line $\Re z = \frac{\mu}{2}$. The fact that the two matrix factors are adjoints of each other originates in the *symmetry* $m^{(0)}(z)[m^{(0)}(\bar{z})]^* = I$ in the inverse NLS scattering problem. Due to this symmetry, it is sufficient to specify the jump matrices only in the upper half-plane and on the real axis.

We perform the following transformations:

- The functions $r$ and $\bar{r}$ are defined on the real axis and have analytic continuations into the complex plane except the points where they have poles. We will utilize the extension of $r$ to the upper half-plane (left of the contour) and of $\bar{r}$ to the lower half-plane (right of the contour).
- By the rules of contour deformation, the right factor of the jump matrix $V^{(0)}$ develops its own contour that splits off to the left of the half-axis $(-\infty, \frac{\mu}{2})$ into the upper half-plane; by symmetry, the left factor splits off to the right, into the lower half-plane. We label the two contours $\Sigma^+$ and $\Sigma^-$, respectively, and we label their union $\Sigma$.
- We let the deforming contours pass through all the points of discrete spectrum $z_k$ in the upper half-plane and through all the corresponding points in the lower half-plane, eliminating the circular contours $C_k$ around individual eigenvalues $z_k$ in the process. The elimination occurs because the difference of the jump matrix on each circle from the jump matrix of the deforming contour has an analytic continuation inside the circle as seen from (1.8).
- We reverse the contour orientation in $\Sigma^+$; thus $\Sigma$ starts at $-\infty$, proceeds through the lower half-plane to $\frac{\mu}{2}$, and returns to $-\infty$ through the upper half-plane; the effect of the reversal of orientation of $\Sigma^+$ on its jump matrix is that it is replaced by its inverse $\left(\begin{smallmatrix} 1 & 0 \\ -r & 1 \end{smallmatrix}\right)$. With the new contour orientation, the symmetry between jump matrices in the upper $(V^+)$ and lower $(V^-)$ complex half-planes is

$$V^-(V^+)^* = I\,. \tag{2.6}$$

*Remark* (Notation). We use $(\pm)$ as *upper indices* to indicate a jump matrix or the component of a contour on the upper or lower complex half-plane. We use $(\pm)$ as

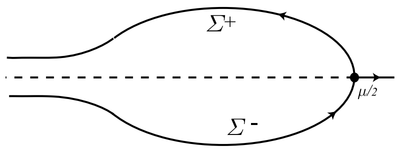


FIGURE 2.2. Contour $\Sigma^{(1)}$.

*lower indices* to indicate the values of a function along the left (+) or right (−) side of the contour.

We come to the RHP $P^{(1)}$, equivalent to RHP $P^{(0)}$, with contour $\Sigma^{(1)} = \Sigma \cup \{z \in \mathbb{R} : z \geq \frac{\mu}{2}\}$, $\Sigma$ oriented counterclockwise, and $\{z \in \mathbb{R} : z \geq \frac{\mu}{2}\}$ oriented from left to right; see Figure 2.2.

$$\text{RHP } P^{(1)} : \begin{cases} m^{(1)}(z)m^{(1)*}(\bar{z}) = I, & z \notin \Sigma^{(1)},\ m^{(1)} \to I \text{ as } z \to \infty, \\ m_+^{(1)} = m_-^{(1)} V^{(1)}, & z \in \Sigma^{(1)}, \end{cases} \tag{2.7}$$

with

$$V^{(1)} = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ -r & 1 \end{smallmatrix}\right) & \text{for } z \in \Sigma^+ \\ \left(\begin{smallmatrix} 1 & \bar{r} \\ 0 & 1 \end{smallmatrix}\right) & \text{for } z \in \Sigma^- \\ V^{(0)} & \text{for } z > \frac{\mu}{2}, \end{cases} \tag{2.8}$$

where by $r = r(z)$ on $\Sigma^+$ we mean the analytic continuation of the function $r(z, x, t, \varepsilon)$ from the half-axis $(-\infty, \frac{\mu}{2})$ into the upper complex half-plane.

*Remark* (Symmetry). The contours $\Sigma$ of all RHP considered henceforth satisfy the condition $\Sigma^- = \overline{\Sigma^+}$, the jump matrices $V$ satisfy the symmetry condition (2.6), $V^- V^{+*} = I$, and the solution matrices $m$ satisfy the symmetry condition (2.7), $m(z)m^*(\bar{z}) = I$, when $z$ is off the contour.

## 2.3 Modification of the Initial Value Problem: Replacement of the Initial Data by Its Leading Asymptotic, the Riemann-Hilbert Problem $P^{(2)}$

The asymptotic analysis of the function $r(z, \varepsilon) = r(z, \varepsilon; x, t)$ is carried out in detail in the appendix. The main tool is Stirling's formula for the asymptotic evaluation of the gamma functions. We obtain

$$r(z, \varepsilon) \sim \begin{cases} e^{-\frac{2i}{\varepsilon} f(z,\varepsilon)} & \text{when } z < \frac{\mu}{2} \\ e^{-\frac{2i}{\varepsilon}(f(z,\varepsilon) + 2\pi i(\frac{\mu}{2} - z))} & \text{when } z > \frac{\mu}{2} \end{cases} \quad \text{as } \varepsilon \to 0 \tag{2.9}$$

with

$$f(z,\varepsilon;x,t)=\Big(\frac{\mu}{2}-z\Big)\Big[\frac{i\pi}{2}+\ln\Big(\frac{\mu}{2}-z\Big)\Big]+\frac{z+T}{2}\ln(z+T)$$
$$+\frac{z-T}{2}\ln(z-T)-T\tanh^{-1}\frac{T}{\mu/2} \tag{2.10}$$
$$-xz-2tz^2+\frac{\mu}{2}\ln 2+\frac{\pi}{2}\varepsilon \quad \text{when } \Im z\geq 0\,,$$

where positive values have real logarithms and $f=f(z,\varepsilon;x,t)$ is analytic in the upper complex half-plane (minus the imaginary segment $[0,T]$ when $\mu<2$). The function $f$ is defined on the open lower complex half-plane by Schwarz reflection.

Our approach, henceforth, is that we *redefine* the initial data to have a reflection coefficient given by the right-hand side of (2.9). Corollary 4.8 shows that the new initial data agree to the order $O(\varepsilon)$ with the original initial data $q(x,0,\varepsilon)$ as $\varepsilon\to 0$.

The RHP $P^{(2)}$, still on contour $\Sigma^{(2)}=\Sigma^{(1)}$ resulting from the above approximation, is

$$\text{RHP } P^{(2)}: m_+^{(2)}=m_-^{(2)}V^{(2)}\,,\quad m\to\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad \text{as } z\to\infty\,, \tag{2.11}$$

where

$$V^{(2)}=\begin{pmatrix}1&0\\-e^{-2if(z)/\varepsilon}&1\end{pmatrix}\ \text{for } z\in\Sigma^+\,,\quad V^{(2)}=V^{(0)}\ \text{for } z>\frac{\mu}{2}\,. \tag{2.12}$$

## 2.4 The $g$-Function Mechanism: The Riemann-Hilbert Problem $P^{(3)}$

The goal of this step is to transform the RHP $P^{(2)}$ to a RHP with a jump matrix that is piecewise constant in $z$ in the limit $\varepsilon\to 0$. We introduce

$$m^{(3)}=m^{(2)}\begin{pmatrix}e^{\frac{2i}{\varepsilon}g(z)}&0\\0&e^{-\frac{2i}{\varepsilon}g(z)}\end{pmatrix}, \tag{2.13}$$

where the analytic in $\bar{\mathbb{C}}\setminus\Sigma^{(2)}$, complex-valued function $g(z;x,t)$ is to be determined.

Preserving the symmetry $m^{(3)}(z)m^{(3)}(\bar z)^*=I$ translates directly to the requirement of Schwarz reflection invariance $g(\bar z)=\overline{g(z)}$ of $g$.

The RHP $P^{(3)}$ on the contour $\Sigma^{(3)}=\Sigma^{(2)}=\Sigma^{(1)}$ is

$$\text{RHP } P^{(3)}: m_+^{(3)}=m_-^{(3)}V^{(3)}\,,\quad m^{(3)}(\infty)=e^{\frac{2i}{\varepsilon}g(\infty)\sigma_3}\,,$$
$$\text{where } \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{2.14}$$

$$V^{(3)}\big|_{z\in\Sigma^+}=\begin{pmatrix}e^{\frac{2i}{\varepsilon}(g_+-g_-)}&0\\-e^{\frac{2i}{\varepsilon}(g_++g_--f)}&e^{-\frac{2i}{\varepsilon}(g_+-g_-)}\end{pmatrix},$$
$$V^{(3)}\big|_{z\in(\frac{\mu}{2},\infty)}=\begin{pmatrix}(1+e^{-\frac{8\pi}{\varepsilon}(z-\frac{\mu}{2})})e^{\frac{2i}{\varepsilon}(g_+-g_-)}&0\\e^{-\frac{4\pi}{\varepsilon}(z-\frac{\mu}{2})+\frac{2i}{\varepsilon}(g_+-g_--f)}&e^{-\frac{2i}{\varepsilon}(g_+-g_-)}\end{pmatrix}. \tag{2.15}$$

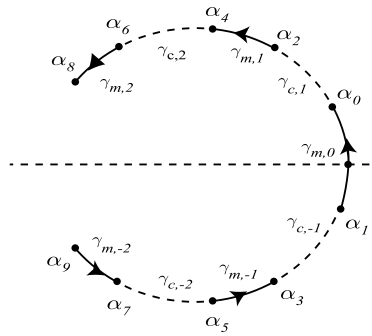


FIGURE 2.3. Main and complementary arcs.

The eventual asymptotic reduction of this matrix to a piecewise constant matrix is achieved through two types of factorization given by the formulae

$$\begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -ab^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b^{-1} \\ 0 & 1 \end{pmatrix} \tag{2.16}$$

and

$$\begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ab & 1 \end{pmatrix}, \tag{2.17}$$

where $a$ and $b$ are the (11) and (21) elements of the jump matrix. The strategy is to define on the contour $\Sigma$ a set of *main arcs* (full lines, Figure 2.3) and a set of interlacing *complementary arcs* (dashed line) and apply the first factorization to the former and the second to the latter. To achieve an asymptotically piecewise constant jump matrix, we require on the upper half-plane

$$\begin{aligned} &\text{on main arcs: } b = \text{const}, \begin{cases} ab^{-1} \to 0 & \text{right of contour} \\ a^{-1}b^{-1} \to 0 & \text{left of contour} \end{cases} \\ &\text{on complementary arcs: } a = \text{const}, \begin{cases} \text{either } a^{-1}b \to 0 & \text{right of contour} \\ \text{or } ab \to 0 & \text{left of contour.} \end{cases} \end{aligned} \tag{2.18}$$

Inserting what $a$ and $b$ stand for into the jump matrix for $\Sigma$, we obtain directly

$$\begin{aligned} &\text{on main arcs: } g_+ + g_- - f = W, \begin{cases} \Im(2g_- - f) < 0 & \text{right of contour} \\ \Im(2g_+ - f) < 0 & \text{left of contour} \end{cases} \\ &\text{on complementary arcs: } g_+ - g_- = \Omega, \\ &\qquad \begin{cases} \text{either } \Im(2g_- - f) > 0 & \text{right of contour} \\ \text{or } \Im(2g_+ - f) > 0 & \text{left of contour,} \end{cases} \end{aligned} \tag{2.19}$$

where $W$ and $\Omega$ are *real* constants (independent of $z$ but dependent on $x$ and $t$). Of course, $W$ may take on different values on different main arcs and the analogous statement is true for $\Omega$. We require the reality of $W$ and $\Omega$ so that the jump matrix $V^{(3)}$ remains bounded when $\varepsilon$ approaches zero. The above relations suggest introducing the function

$$h(z) = 2g(z) - f(z) \tag{2.20}$$

that is analytic in the intersections of the domains of analyticity of $g$ and $f$. Then

$$\begin{aligned} &\text{on main arcs: } h_+ + h_- = 2W, \begin{cases} \Im h_- < 0 & \text{right of contour} \\ \Im h_+ < 0 & \text{left of contour} \end{cases} \\ &\text{on complementary arcs: } h_+ - h_- = 2\Omega, \\ &\qquad\qquad \begin{cases} \text{either } \Im h_- > 0 & \text{right of contour} \\ \text{or } \Im h_+ > 0 & \text{left of contour.} \end{cases} \end{aligned} \tag{2.21}$$

We will return to these relations after we have made a precise definition of the partitioning of the contour. This partitioning includes an *arc to infinity* defined below, on which factorization (2.17) and conditions (2.21) apply, and in addition $\Omega = 0$. Although we do not label this arc as complementary, it does behave as a complementary arc with $\Omega = 0$.

In the upper complex half-plane, we postulate a finite sequence of points, $\alpha_0$, $\alpha_2, \dots, \alpha_{4N}$, that partition $\Sigma^+$, the upper half-plane part of contour $\Sigma$, to a set of arcs (see Figure 2.3)

$$\Sigma^+ = \left(\tfrac{\mu}{2}, \alpha_0\right), (a_0, \alpha_2), (a_2, \alpha_4), \dots, (\alpha_{4N}, -\infty) . \tag{2.22}$$

We define the set of main arcs and interlace with them the set of complementary arcs and the arc to infinity as follows:

$$\begin{aligned} &\text{main arcs: } \gamma_m^+ = \left(\tfrac{\mu}{2}, \alpha_0\right) \cup (\alpha_2, \alpha_4) \cup \dots \cup (\alpha_{4N-2}, \alpha_{4N}) , \\ &\text{complementary arcs: } \gamma_c^+ = (\alpha_0, \alpha_2) \cup (\alpha_4, \alpha_6) \cup \dots \cup (\alpha_{4N-4}, \alpha_{4N-2}) , \\ &\text{arc to infinity: } \gamma_\infty^+ = (\alpha_{4N}, -\infty) . \end{aligned} \tag{2.23}$$

We label the $N + 1$ main arcs $\gamma_{m,0}^+ = (\frac{\mu}{2}, \alpha_0), \dots, \gamma_{m,N}^+ = (\alpha_{4N-2}, \alpha_{4N})$; we label the $N$ complementary arcs $\gamma_{c,1}^+ = (\alpha_0, \alpha_2), \dots, \gamma_{c,N}^+ = (\alpha_{4N-4}, \alpha_{4N-2})$. We also label the union of the main and complementary arcs in the upper half-plane,

$$\tilde{\gamma}^+ = \gamma_m^+ \cup \gamma_c^+ , \quad \text{hence} \quad \Sigma^+ = \tilde{\gamma}^+ \cup \gamma_\infty^+ . \tag{2.24}$$

The complex conjugates of the points $\alpha_{2k}$ are labeled $\alpha_{2k+1} = \bar{\alpha}_{2k}$, $k = 0, 1, \dots, 2N$; replacing the upper plus by a minus in any of the above arcs indicates the complex conjugates of that arc; finally, the absence of a plus or minus indicates the union of the two, e.g., $\tilde{\gamma} = \tilde{\gamma}^+ \cup \tilde{\gamma}^-$. All arcs of the upper and lower half-planes inherit the orientation of $\Sigma$. (The arcs $\gamma_{m,j}^-$ and $\gamma_{c,j}^-$ coincide with the arcs $\gamma_{m,-j}$ and $\gamma_{c,-j}$ on Figure 1.1, $j = 1, 2, \dots, N$.)

On the main arcs, the reality of $W$ and the fact that $\Im h$ must be negative on both sides imply that $\Im h = 0$. On the complementary arcs, $\Im h$ must be positive on at least one side. If $\Im h$ is positive on both sides of the contour, it is conceivable

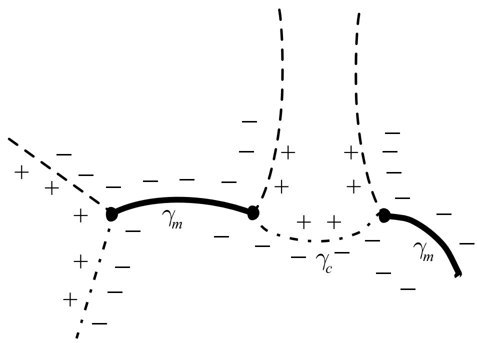

FIGURE 2.4. Signs of $\Im h$.

that the contour may be deformed to the boundary of the set $\{z : \Im h(z) > 0\}$ where again $\Im h = 0$. Our analysis below shows that this is possible on the complementary arcs, but not possible on the arc to infinity $\gamma_\infty$. Summarizing this together with (2.21), we require the following zero behavior and sign structure of $\Im h$ on and near the contour (see Figure 2.4):

$$\Im h(z) = 0 \text{ when } z \in \tilde{\gamma}^+, \quad \Im h(z) > 0 \text{ when } z \in \gamma_\infty^+, \tag{2.25}$$

$$\begin{cases} \text{left and right of } \gamma_m^+: & \Im h(z) < 0, \\ \text{across } \gamma_c^+: & \Im h(z) \text{ changes sign}, \\ \text{on } \gamma_\infty^+: & \Im h(z) > 0. \end{cases} \tag{2.26}$$

On the upper half-plane, in the configuration that achieves the desired sign structure, we will see that the curve $\Im h = 0$ starts at point $\frac{\mu}{2}$ on the real axis and follows the contour $\Sigma^+$ to the point $\alpha_{4N}$ (point $\alpha_8$ in the figure); from there it separates from the contour and connects to point $-\frac{\mu}{2}$ back on the real axis.

We *normalize* $g$ (and $h$) up to a constant by selecting the value of $W$ on the main arc $\gamma_{m,0}$ to be $W_0 = 0$. Thus, $W$ and $\Omega$ now take constant real values $W_i$ and $\Omega_i$ on the intervals $\gamma_{m,i}^+$ and $\gamma_{c,i}^+$, $i = 1, \dots N$. We consider $W$ and $\Omega$ as two $N$-dimensional real vectors with components $W_i$ and $\Omega_i$, $i = 1, \dots N$.

The simplicity of the conditions on $h$ versus the conditions on $g$ is, of course, balanced by the fact that $h$ has a jump across the real axis equal to the jump of $-f$, while $g$ is analytic on $\mathbb{R}$ except at $\frac{\mu}{2}$.

## 2.5 Topology of the Zero Level Curve of $\Im h$ and Genus of $q_0(x, t, \varepsilon)$

The topology of the zero level curve of $\Im h$ can be determined based on the following observations:

- All branches of $\Im h = 0$ in the upper complex half-plane begin and end at either a branch point $\alpha_{2k}$, or on the real axis, or at infinity (in the case $\mu < 2$ they can also originate from the segment $[0, T]$).

- As the construction of the function $h$ in the following section shows, there are three branches of $\Im h = 0$ emanating from each simple branch point $\alpha_{2k}$ (see (3.10)).
- On the real axis, $\Im h = -\Im f$, since $g$ is required to be real on $\mathbb{R}$. Thus, direct study of $\Im f$ shows that $\Im h$ has two zeros $z = \frac{\mu}{2}$ and $z = -\frac{\mu}{2}$ on the real axis.
- At infinity, the branches of $\Im h = 0$ are asymptotic to the ones of $f(z)$ since $g(z)$ approaches a real constant as $z \to \infty$.
- There cannot be a closed, bounded loop of zero level curve of $\Im h = 0$ if $\Im h$ has no singularities, i.e., is harmonic, inside the loop, since this would imply that $h$ is identically zero by the maximum modulus theorem.

Based on these observations, we determine that conditions (2.21) can be satisfied only in the following cases:

- When $t = 0$, there is exactly one branch of $\Im f = 0$ going to infinity. Thus, at $t = 0$, there can be only one branch point. Three branches of $\Im h = 0$ will emanate from it, one going to infinity and two to the points $z = \frac{\mu}{2}$ and $z = -\frac{\mu}{2}$. Thus there is a curve connecting these two points on which $\Im h = 0$.
- When $t > 0$, there are exactly three branches of $\Im f = 0$ going to infinity. This allows either one or three branch points, provided that in the case $\mu < 2$ no zero level curves of $\Im h$ emanate from $[0, T]$. In the case of one branch point, there will be a branch of $\Im h = 0$ from infinity to infinity in addition to the branch structure of $t = 0$. In the case of three simple branch points, there will be a zero level curve of $\Im h$ connecting $\frac{\mu}{2}$ to $-\frac{\mu}{2}$ that passes through all three points. Two of the three zero level curves emanating from each of the three branch points will lie on on this curve, while the remaining one will go to infinity. In the pure radiation case $\mu \geq 2$, there cannot be more than three branch points, since this would require more connections to infinity; thus the *genus in the pure radiation case cannot exceed* 2. When $\mu < 2$, connections to $[0, T]$ are possible, so the question of a bound for the genus of the soliton case remains open at the moment.

Motivated by the above considerations, we prefer to think in terms of contour

$$\gamma^+ = \tilde{\gamma}^+ \cup \gamma^+_{\text{con}} \tag{2.27}$$

with corresponding definitions for $\gamma^-$ and $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^+_{\text{con}}$ is a zero level curve of $\Im h$ that connects $\alpha_{4N}$ with $-\frac{\mu}{2}$. Thus, $\gamma^+$ connects $\frac{\mu}{2}$ to $-\frac{\mu}{2}$ and $\gamma = \gamma^+ \cup \gamma^-$ is a closed curve. The conditions

$$\Im h(z) = 0 \quad \text{when } z \in \gamma\,, \tag{2.28}$$

$$\begin{cases} \text{left and right of } \gamma^+_m: & \Im h(z) < 0, \\ \text{across } \gamma^+ \setminus \gamma^+_m = \gamma^+_c \cup \gamma^+_{\text{con}}: & \Im h(z) \text{ changes sign,} \end{cases} \tag{2.29}$$

imply the earlier ones (2.25) and (2.26). Indeed, if the new conditions are satisfied, $\gamma_{\mathrm{con}}$ must have $\Im h > 0$ on its right, because this holds on the upper lip of $\mathbb{R}$ to the left of $z = \frac{\mu}{2}$. Thus, $\gamma_\infty$ satisfying the earlier conditions can be defined as a small deformation of $(-\infty, -\frac{\mu}{2}) \cup \gamma_{\mathrm{con}}$. We adopt the new conditions as our requirement.

## 2.6 Outline of Results on *h* and *α*

The construction of the functions $g$ and $h$ is outlined in Section 3. A key element that enters the expression for $g$ and $h$ and leads to the solution $q_0(x, t, \varepsilon)$ of NLS is the radical $R(z) = \prod_{k=0}^{4N+2} \sqrt{(z - \alpha_k)}$ with branch cut $\gamma_m$ and the corresponding Riemann surface $\mathcal{R}(x, t)$. For each $N$, we derive a system of nonlinear equations (3.5) and (3.9)

$$F(\alpha, x, t) = F_N(\alpha, x, t) = 0 \tag{2.30}$$

for the $4N + 2$ unknowns $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{4N+1})$.

The evolution and degeneracy theorems (see below) are our main tools to keep track of the time evolution of the Riemann surface $\mathcal{R}(x, t)$ associated with the solution. The solution theorem describes sufficient conditions that allow us to reconstruct $q_0(x, t, \varepsilon)$ through $\mathcal{R}(x, t)$.

### Prebreak

The main result here is the construction of the function $h(z; x, t)$ (see Section 4.2) that satisfies (2.21) with genus 0 ($N = 0$) when the $(x, t)$ are in a space-time region $0 \le t < t_0(x) < \infty$. System (2.30) has a single unknown, $\alpha_0(x, t) = \alpha = a + ib$, in the upper half-plane and reduces to a pair of real equations from which $a$ and $b$ are found. For all $t > 0$, there are two solution branches of these equations. Exactly one of these connects smoothly to the initial ($t = 0$) values of $a = a(x, 0) = \frac{\mu}{2} \tanh x$ and $b = b(x, 0) = \operatorname{sech} x$. The connecting solution branch exists uniquely at all times for each $x \neq 0$. Breaking for $x \neq 0$ occurs at $t = t_0(x) < \infty$ because of a breakdown of the required sign structure. At $x = 0$, $t = \frac{1}{2(\mu+2)} = t_0(0)$, uniqueness is lost when the two branches yield the same $\alpha$. In Sections 4.5 and 4.6 we show that for $0 \le t < t_0$ there is a zero level curve of $\Im h$ in the upper half-plane connecting point $\frac{\mu}{2}$ to point $-\frac{\mu}{2}$, passing through the point $\alpha$, and displaying the required sign structure (2.29). This is our contour $\gamma^+$; see Figure 2.5.

### Breaking

Breaking occurs for a topological reason. For $x > 0$ fixed, and as time, increasing from 0, reaches the value $t = t_0(x)$, the curve $\gamma^+$ comes into contact with a second branch of $\Im h = 0$ at a point $z_0$. Necessarily $h'(z_0) = 0$; see Figure 2.6.

As $t$ increases further, the four zero level curves of $\Im h(z; x, t) = 0$ emanating from $z_0$ as a result of the quadratic behavior of $h$ at this point interchange connections; a zero $\Im h$ level curve connection between points $\frac{\mu}{2}$ and $-\frac{\mu}{2}$ ceases to exist.

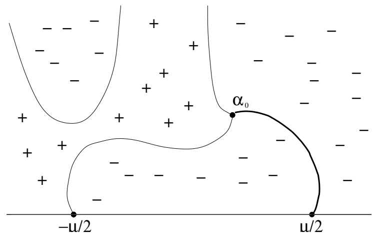


FIGURE 2.5. Zero level curves of $\Im h$, prebreak.

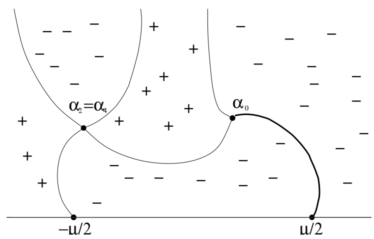


FIGURE 2.6. Zero level curves of $\Im h$, breaking point.

The breaking curve $t = t_0(x)$ is obtained in principle by eliminating $z$ from the system of the three real equations $\Im h(z; x, t) = 0$ and $h'(z; x, t) = 0$. We have analytic formulae for the functions $h(z, x, t)$ and $h'(z, x, t)$, yet the above elimination of $z$ cannot be performed explicitly. We prove the existence of the breaking curve $t_0(x)$ (see Section 5.1), and we calculate explicitly its asymptotic behavior as $x \to \infty$ and $x \to 0^+$ (see Section 5.3).

When the point $(x, t)$ is on the breaking curve, the genus 0 solution breaks down; the sign structure (2.29) on the part of the contour that connects $\alpha_0$ to $-\frac{\mu}{2}$ is violated at the zero of $h'$, i.e., at $z_0$. A degenerate genus 2 ($N = 1$) solution is obtained by identifying the point of contact $z_0$ of the two zero level curves of $\Im h$ with a double point $z_0 = \alpha_2 = \alpha_4$. As described in Theorem 3.1 below, the function $h$, as a degenerate genus 2 solution of the scalar RHP, is identical to function $h$ as a genus 0 solution of the scalar RHP. This extends the region of our solution to $t \le t_0(x)$.

In the case $x = 0$, and only in this case, we have higher degeneracy at $t_0 = \frac{1}{2(\mu+2)}$ and $z_0 = i\sqrt{\mu + 2}$. Then system (2.30) is satisfied with $\alpha_0 = \alpha_2 = \alpha_4 = z_0$.

### Postbreak Local Calculation: $(t - t_0)$ Small

As discussed above, we can solve (2.30) for genus 2 exactly on the breaking curve. We now show that we can solve the system for $N = 1$ in a vicinity of the breaking curve on either side of it (both pre- and postbreak). We face the difficulty that the Jacobian $\frac{\partial F}{\partial \alpha}$ contains all factors of type $\alpha_k - \alpha_l$ and thus vanishes on the breaking curve on which $\alpha_2 = \alpha_4$. To overcome this, we make a change of the

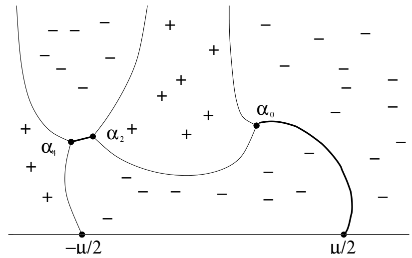


FIGURE 2.7. Zero level curves of $\Im h$, postbreak.

variables $\alpha$ (see Section 6.2, a similar change of variable was made in [24]) so that the new Jacobian is different from zero in a neighborhood of the breaking curve. We then use the implicit function theorem to show the local solvability of (2.30) near the breaking curve for the three unknowns $\alpha_0$, $\alpha_2$, and $\alpha_4$.

The sign of $\Im h$ near the subcontour $\gamma_{m,1}^{+}$, connecting $\alpha_2$ and $\alpha_4$, changes as $t$ crosses the value $t = t_0$. This sign cannot satisfy (2.29) in the prebreak region, for otherwise conditions (2.19) would be satisfied for $N = 0$ and $N = 1$ and we would be able to construct two different asymptotic behaviors for NLS. Thus, this sign satisfies (2.29) above the breaking curve (postbreak region), and our solution, now of genus 2, extends locally in a region above the breaking curve. The topology of zero level curves of $\Im h$ in the postbreak region is shown in Figure 2.7.

## Postbreak Global Calculation

In the prebreak region, $\alpha_0$ is found implicitly as the solution to some transcendental equations. The leading asymptotic behavior of the solution of NLS in this region is given in terms of the $\alpha_0$. For $\mu = 2$ and $\mu = 0$ expressions for $\alpha_0$ are explicit (see (4.29)–(4.30)).

By the evolution theorem (Theorem 3.2), the existence of nondegenerate $\alpha$ for some $(x_0, t_0)$ implies the existence of a nondegenerate $\alpha$ in a neighborhood of $(x_0, t_0)$ (if all $\alpha_{2k}$ are distinct, then the Jacobian matrix $|\frac{\partial F}{\partial \alpha}| \neq 0$). Studying global solvability in the postbreak region, i.e., solvability for all (or almost all) $(x, t)$ in that region, includes two crucial elements: (1) existence of a solution to (2.30) for at least one point $(x_0, t_0)$ with a nondegenerate $\alpha$, and (2) control over the degeneracy of $\alpha$ together with control over new breaks through collision of different branches of the zero level curve $\Im h = 0$ and through intersection of $\gamma$ with singularities of $f(z)$.

In this paper, we are able to establish the global postbreak solvability of (2.30) for genus 2 ($N = 1$) for the solitonless case $\mu \geq 2$ (see Section 6.4) and to produce the leading asymptotic behavior of the solution of NLS in this region in terms of the $\alpha$ (see Sections 7 and 8). Global postbreak solvability of (2.30) for genus 2 is achieved because we have control over the branches of the zero level curve $\Im h = 0$ (its branches can emanate only from the real axis and from infinity); we use this

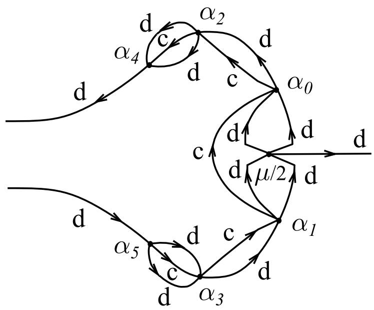


FIGURE 2.8. Contour $\Sigma^{(4)}$. The letter $d$ denotes contours with jump matrix approaching $I$, whereas a $c$ denote contours with jump matrix approaching a constant limit.

to show that the genus does not increase from 2, and that the global solvability of (2.30) implies that conditions (2.19) can be satisfied everywhere above the breaking curve for $N = 1$. In the soliton ($\mu < 2$) case, we can extend the solvability of (2.30) and calculate the leading-order term $q_0(x,t,\varepsilon)$ of $q(x,t,\varepsilon)$ above the breaking curve as long as different branches of $\Im h = 0$ do not collide with each other or with the segment $[0, T]$, i.e., as long as the genus is equal to 2. However, we did not prove solvability for all positive $t$, i.e., global solvability, in this case.

## 2.7 The Riemann-Hilbert Problem $P^{(4)}$ and the Model Riemann-Hilbert Problem $P^{(\mathrm{mod})}$

### Riemann-Hilbert Problem $P^{(4)}$

Following the construction of the function $g$, we transform RHP $P^{(3)}$ to RHP $P^{(4)}$ by performing the contour deformations of Figure 2.8.

- On each arc of the contour $\gamma_m^+$, the left and right factors of (2.16) split off on their own contours to the right and left, respectively, leaving us with three contours, the middle one with the constant jump matrix of the middle factor
$$\begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}W} \\ -e^{\frac{2i}{\varepsilon}W} & 0 \end{pmatrix}$$
and the other two converging exponentially to the identity. The convergence is *uniform* if a neighborhood of the endpoints is excluded. On the arc $(\alpha_0, \bar{\alpha}_0)$, we make an extra deformation; the contour of constant jump is moved to the left of the left contour whose jump matrix thus suffers a conjugation with the constant jump matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (recall $W_0 = 0$), but remains exponentially converging to the identity. Finally, the contour near $\frac{\mu}{2}$ is a subset of the real axis: the lines emanating from $\frac{\mu}{2}$ are shown slanted only to describe the deformation; they lie exactly on the $x$-axis on some neighborhood of $\frac{\mu}{2}$ before leaving the real axis to connect to $\alpha_0$ or $\alpha_1$.

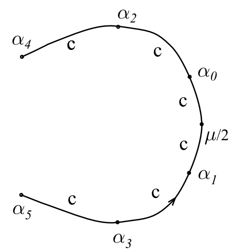


FIGURE 2.9. Contour $\Sigma^{(\mathrm{mod})} = \Sigma \setminus \gamma_\infty = \tilde{\gamma}$.

- On each arc of the contour $\gamma_c^+$ one of the factors (2.17) with 1s on the diagonal splits off on its own contour on the side where $\Im h > 0$, leaving us with two contours, this one with a jump that converges to the identity, and one with the constant jump
$$\begin{pmatrix} e^{\frac{2i\Omega}{\varepsilon}} & 0 \\ 0 & e^{-\frac{2i\Omega}{\varepsilon}} \end{pmatrix}.$$
The convergence is again uniform if neighborhoods of the endpoints are omitted.
- The arc $\gamma_\infty$ that connects $\alpha_{4N}$ to $-\infty$ has a jump matrix that converges exponentially to the identity matrix uniformly outside any neighborhood of $\alpha_{4N}$.

### The Model Riemann-Hilbert Problem $P^{(\mathrm{mod})}$: Genus $2N$

In Section 7 we derive an explicit formula for the solution of the *model* RHP obtained from $P^{(4)}$ when the jump matrices that converge to the identity as $\varepsilon \to 0$ are neglected (replaced with the identity). We have

$$\text{RHP } P^{(\mathrm{mod})}: \quad m_+^{(\mathrm{mod})} = m_-^{(\mathrm{mod})} V^{(\mathrm{mod})} \quad \text{when } z \in \Sigma^{(\mathrm{mod})} \tag{2.31}$$

on the contour $\Sigma^{(\mathrm{mod})} = \tilde{\gamma} = \Sigma \setminus \gamma_\infty$, which has the usual symmetry (see Figure 2.9). The piecewise constant jump matrix

$$V^{(\mathrm{mod})} = \begin{cases} \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}W} \\ -e^{\frac{2i}{\varepsilon}W} & 0 \end{pmatrix} & \text{when } z \in \gamma_m^+ \\ \begin{pmatrix} e^{\frac{2i}{\varepsilon}\Omega} & 0 \\ 0 & e^{-\frac{2i}{\varepsilon}\Omega} \end{pmatrix} & \text{when } z \in \Sigma^{(\mathrm{mod})+} \setminus \gamma_m^+. \end{cases} \tag{2.32}$$

is the asymptotic limit of $V^{(4)}$ as $\varepsilon \to 0$.

The introduction of $g$ in (2.13) changes the normalization of $m^{(3)}, m^{(4)}$, and $m^{(\mathrm{mod})}$ to

$$m^{(3)}, m^{(4)}, m^{(\mathrm{mod})} \to \begin{pmatrix} e^{\frac{2i}{\varepsilon}g(\infty)} & 0 \\ 0 & e^{-\frac{2i}{\varepsilon}g(\infty)} \end{pmatrix} \quad \text{as } z \to \infty\,. \tag{2.33}$$

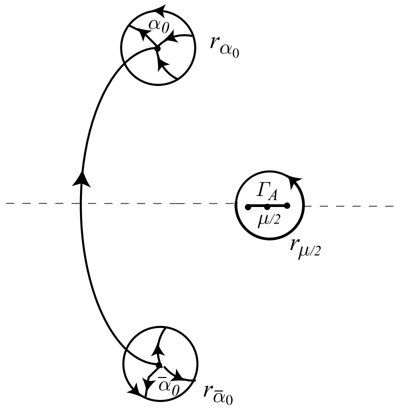


FIGURE 2.10. Contour $\Sigma^{(\mathrm{app})}$.

The model RHP is solved by an explicit formula for any genus in Section 7.

## 2.8 Solution of NLS with Error Estimate

We derive an *error estimate* that proves that replacing $P^{(4)}$ with the model problem produces the required solution $q_0(x, t, \varepsilon)$ of NLS. We treat the case of genus 0 ($N = 0$); the same derivation applies to the case of higher genus. The jump matrix of the model problem is the limit of the jump matrix $V^{(4)}$ as $\varepsilon \to 0$. Ideally, we would like to set $m^{(4)} = m^{(\mathrm{err})} m^{(\mathrm{mod})}$ so it would peel off the solution $m^{(\mathrm{mod})}$ of the model problem and leave us with an RHP for some $m^{(\mathrm{err})}$; it would represent our error, for which we would seek an estimate. In reality, the RHP for $m^{(\mathrm{err})}$ is then too singular at the branch points $\alpha$ and at $\frac{\mu}{2}$. We note that near these points there is no uniform convergence $V^{(4)} \to I$ as $\varepsilon \to 0$. To overcome this difficulty, we treat these points specially. Instead of peeling off $m^{(\mathrm{mod})}$, we peel off a modified matrix $m^{(\mathrm{app})}$ defined in the following way:

(1) $m^{(\mathrm{app})}(z)$ equals the solution of the model problem $m^{(\mathrm{mod})}(z)$ outside three circles, $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $r_{\mu/2}$, centered at the points $\alpha_0$, $\bar{\alpha}_0$, and $\frac{\mu}{2}$, respectively, with radii $2\delta$, where $\delta$ is positive and small but independent of $\varepsilon$; see Figure 2.10.

(2) $m^{(\mathrm{app})}(z)$ is a parametrix of $V^{(4)}$ inside the circles of $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $|z - \frac{\mu}{2}| = \delta$; i.e., it satisfies the jump conditions of the RHP $P^{(4)}$ inside these circles exactly.

(3) The jump of $m^{(\mathrm{app})}(z)$ across circles $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $r_{\mu/2}$, i.e., across the real intervals $\delta < |z - \frac{\mu}{2}| < 2\delta$, must be of order $I + O(\varepsilon)$ uniformly.

If the above-mentioned matrix $m^{(\mathrm{app})}(z)$ exists, it should satisfy the RHP $P^{(\mathrm{app})} = (V^{(\mathrm{app})}, \Sigma^{(\mathrm{app})})$, where $\Sigma^{(\mathrm{app})} = \Sigma^{(\mathrm{mod})}$, $V^{(\mathrm{app})} = V^{(\mathrm{mod})}$ outside the three circles $r$, and $\Sigma^{(\mathrm{app})} = \Sigma^{(4)}$ and $V^{(\mathrm{app})} = V^{(4)}$ inside the circles $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $|z - \frac{\mu}{2}| = \delta$; see Figure 2.10.

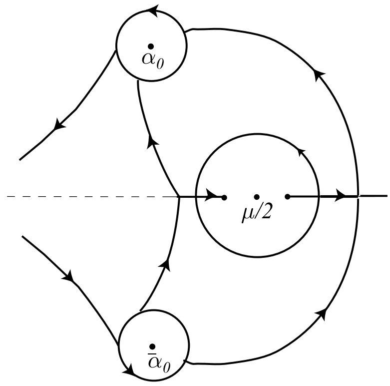


FIGURE 2.11. Contour $\Sigma^{(\text{err})}$.

Peeling off $m^{(\text{app})}$ through $m^{(4)} = m^{(\text{err})}m^{(\text{app})}$ corresponds to the reduction of the RHP $P^{(4)}$ to the RHP $P^{(\text{err})} = (V^{(\text{err})}, \Sigma^{(\text{err})})$, where $\Sigma^{(\text{err})}$ is the union of all solid lines of Figure 2.11. On the circles $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $r_{\mu/2}$ and inside $r_{\mu/2}$ with $|z - \frac{\mu}{2}| > \delta$, we have $V^{(\text{err})} = I + O(\varepsilon)$. On the rest of $\Sigma^{(\text{err})}$ the jump $V^{(\text{err})} = V^{(4)}$.

Parametrices for $m^{(\text{app})}$ in the neighborhood of the points $\alpha_0$ and $\bar{\alpha}_0$ satisfying the above conditions have been constructed through the use of Airy functions; see [10]. We do not repeat this analysis here. We defer the construction of the parametrix $m^{(\text{app})}(z)$ around the point $\frac{\mu}{2}$ to Section 9 and proceed assuming this construction has been completed.

The solution of our initial value problem for NLS is

$$q(x,t,\varepsilon) = -2 \lim_{z\to\infty} z(m^{(2)}(z) - I)_{12}\,, \tag{2.34}$$

the subscripts indicating the (12) matrix entry. Putting together previous reductions of the RHP $P^{(2)}$, we have for $|z|$ large enough

$$m^{(2)}(z) = m^{(4)}(z)e^{-\frac{2i}{\varepsilon}g(z)\sigma_3} = m^{(\text{err})}(z)m^{(\text{mod})}e^{-\frac{2i}{\varepsilon}g(z)\sigma_3} \equiv m^{(\text{err})}(z)M(z)\,, \tag{2.35}$$

where $M(z) = m^{(\text{mod})}e^{-\frac{2i}{\varepsilon}g(z)\sigma_3}$. An easy calculation gives

$$q(x,t,\varepsilon) = -2 \lim_{z\to\infty} z(M(z) - I)_{12} - 2 \lim_{z\to\infty} z(m^{(\text{err})}(z) - I)_{12}\,, \tag{2.36}$$

where the first term is our approximation $q_0(x,t,\varepsilon)$ of the solution $q(x,t,\varepsilon)$ to (1.1) presented in the main theorem. The estimate for the second term,

$$m_1^{(\text{err})} \equiv \lim_{z\to\infty} z(m^{(\text{err})}(z) - I) = O(\varepsilon) \tag{2.37}$$

uniformly in $x$ and $t$ on compact sets away from breaking curves, is proven in Section 9.

*Remark.* A uniform estimate in the neighborhood of the breaking curve is possible that gives an error of order $O(\varepsilon^{1/2})$ but is omitted from this study.

## 3 Construction of the Functions *g* and *h*

*Remark* (Generality of the Initial Data). The procedure in this section applies to any $f(z)$ that is analytic and Schwarz reflection invariant along $\tilde{\gamma}$. For the sake of concrete calculation and precision in stating results, our exposition as well as the results assume that $f(z)$ is given by (2.10).

### 3.1 Formulae for *g* and *h*

The equalities in (2.19) and their differentiated version, together with the requirement that $g(z)$ be analytic at $\infty$, pose a pair of scalar additive RHPs, one for $g(z)$ and the second one for $g'(z)$ on the still unknown contour $\tilde{\gamma}$:

$$\begin{cases} g_+ + g_- = f + W_0 = f & \text{on } \gamma_{m,0}^+ \\ g_+ + g_- = f + W_i & \text{on } \gamma_{m,i}^+ \\ g_+ - g_- = \Omega_i & \text{on } \gamma_{c,i}^+ \end{cases} \quad \text{and} \quad \begin{cases} g'_+ + g'_- = f' & \text{on } \gamma_{m,0}^+ \\ g'_+ + g'_- = f' & \text{on } \gamma_{m,i}^+ \\ g'_+ - g'_- = 0 & \text{on } \gamma_{c,i}^+, \end{cases} \qquad i = 1, 2, \dots, N. \tag{3.1}$$

We recall the normalization $W_0 = 0$. Our construction of a function $g(z)$ that satisfies conditions (2.19) begins with the differentiated problem on an arbitrary, oriented, non-self-intersecting Schwarz-reflection-invariant contour

$$\tilde{\gamma} = \left(\alpha_{4N+1}, \alpha_{4N-1}, \dots, \alpha_1, \frac{\mu}{2}, \alpha_0, \alpha_2, \dots, \alpha_{4N}\right), \tag{3.2}$$

as in Figure 2.3, where the even-indexed $\alpha_{2k}$ are distinct points with $\Im\alpha_{2k} > 0$ and $\alpha_{2k+1} = \bar{\alpha}_{2k}$, $k = 0, 1, \dots, 2N$. The contour $\tilde{\gamma}$ intersects $\mathbb{R}$ only at the point $\frac{\mu}{2}$; we assume that it does not pass through the singularities of $f(z)$. In the case when $\mu < 2$, that means that $\tilde{\gamma}$ does not pass through the segment $[-T, T]$ on the imaginary axis, which is a branch cut of $f(z)$.

One easily verifies that the jump conditions (3.1) of the differentiated problem are satisfied by the expression (see [19, 41])

$$g'(z) = \frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{f'(\zeta)}{(\zeta - z)R_+(\zeta)}\, d\zeta = \frac{R(z)}{4\pi i} \oint_{\hat{\gamma}} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)}\, d\zeta\,, \tag{3.3}$$

where $R(z) = \prod_{k=0}^{4N+2} \sqrt{(z - \alpha_k)}$, and $R_\pm(z)$ denotes the value of $R$ on the left and right side of the branch cut $\gamma_m$; the sign of $R$ is determined by its behavior at infinity, $R(z) \sim -z^{2N+1}$ as $z \to \infty$. The contour $\hat{\gamma}$ in the second expression is the union of a loop that starts at $z = \frac{\mu}{2} - 0$, encircles the contour $\tilde{\gamma}^+$ clockwise, and closes at $z = \frac{\mu}{2} + 0$ together with its complex conjugate, also oriented clockwise. We refer to $\hat{\gamma}$ as the *figure 8*; see Figure 3.1. The point $z$ is *outside* the figure 8.

By an application of residue calculus, we find that the function $h' = 2g' - f'$ is given by the formula

$$h'(z) = \frac{R(z)}{2\pi i} \oint_{\hat{\gamma}} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)}\, d\zeta\,, \tag{3.4}$$

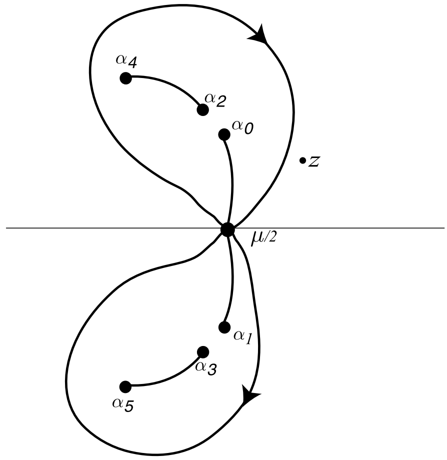


FIGURE 3.1. Contour $\hat{\gamma}$.

where now $z$ is *inside* the figure 8. $f'$ is Schwarz reflection invariant; hence $g'$ is real on $\mathbb{R}$.

The requirement that $g(z)$ be analytic at $\infty$ implies that $g'(z) \sim O(z^{-2})$ as $z \to \infty$; the latter is equivalent to the $2N + 2$ *moment conditions*

$$\text{moment condition } M_k\text{:} \quad \oint_{\hat{\gamma}} \frac{\zeta^k f'(\zeta)}{R_+(\zeta)}\, d\zeta = 0\,, \quad k = 0, 1, \dots, 2N+1, \tag{3.5}$$

that we obtain by expanding $(\zeta - z)^{-1})$ in the integral in powers of $z^{-1}$.

Assuming that the points $\alpha$ are chosen so that the moment conditions and hence the relation $g'(z) \sim O(z^{-2})$ hold, the primitive function

$$g(z) = \int_\infty^z g'(z)dz + g(\infty) \tag{3.6}$$

is analytic, single valued, and Schwarz reflection invariant in $\bar{\mathbb{C}} \setminus \tilde{\gamma}$; hence it is real on $\mathbb{R} \setminus \{\frac{\mu}{2}\}$. The constant of integration $g(\infty)$ is chosen so that

$$\text{at } z = \frac{\mu}{2}, \quad g_+ + g_- - f = W_0 = 0\,, \quad \text{equivalently,} \quad h_+ + h_- = 0\,, \tag{3.7}$$

is satisfied, guaranteeing that the first jump relation (3.1) holds. The constant jumps of $h$, $\Omega_i$, and $W_i$ satisfy the relations

$$\begin{cases} \Omega_{i+1} - \Omega_i = \frac{1}{2}(h_+ - h_-)|_{\alpha_{4i-2}}^{\alpha_{4i}} = \frac{1}{2}\int_{\gamma_{m,i}^+} h'_+(\zeta) - h'_-(\zeta)d\zeta \\ \qquad\qquad = \frac{1}{2}\oint_{\hat{\gamma}_{m,i}^+} h'(\zeta)d\zeta, \quad \Omega_{N+1} = 0, \\ W_i - W_{i-1} = \frac{1}{2}(h_+ + h_-)|_{\alpha_{4i-4}}^{\alpha_{4i-2}} = \frac{1}{2}\int_{\gamma_{c,i}^+} h'_+(\zeta) + h'_-(\zeta)d\zeta \\ \qquad\qquad = \frac{1}{2}\oint_{\hat{\gamma}_{c,i}^+} h'(\zeta)d\zeta, \quad W_0 = 0, \end{cases} \tag{3.8}$$

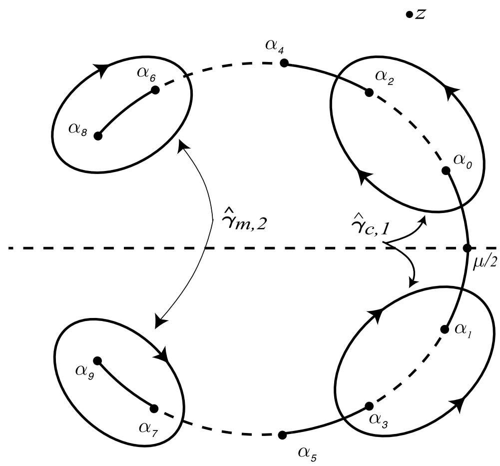


FIGURE 3.2. Contour $\hat{\gamma}$.

where $i = 1, 2, \dots, N$. The contour $\hat{\gamma}^{\pm}_{m,i}$ encircles the arc $\gamma^{\pm}_{m,i}$ clockwise; the contour $\hat{\gamma}^{\pm}_{c,i}$ is a loop consisting of the union of two oppositely oriented arcs as in Figure 3.2, where $\hat{\gamma}_{m,i} = \hat{\gamma}^{+}_{m,i} \cup \hat{\gamma}^{-}_{m,i}$ and $\hat{\gamma}_{c,i} = \hat{\gamma}^{+}_{c,i} \cup \hat{\gamma}^{-}_{c,i}$. (Equivalently, $\hat{\gamma}^{\pm}_{c,i}$ may be taken to be a cycle that closes through the lower sheet of the Riemann surface of $R(z)$.) We will also use the contour $\hat{\gamma}_{m,0}$ that encircles $\gamma_{m,0}$ clockwise and has the shape of the figure 8 with a point of self-contact at $\frac{\mu}{2}$. We make the following important observations:

(1) $\Omega$ and $W$ are calculated in terms of $h'$, that is, in terms of the $\alpha$.
(2) The required reality of $\Omega$ and $W$ is equivalent to the following $2N$ real *integral conditions*:

$$\begin{cases} I_{m,i} : \Im \oint_{\hat{\gamma}^{+}_{m,i}} h'(\zeta)d\zeta = 0 \\ I_{c,i} : \Im \oint_{\hat{\gamma}^{+}_{c,i}} h'(\zeta)d\zeta = 0, \end{cases} \quad \text{equivalently} \quad \begin{cases} \oint_{\hat{\gamma}_{m,i}} h'(\zeta)d\zeta = 0 \\ \oint_{\hat{\gamma}_{c,i}} h'(\zeta)d\zeta = 0, \end{cases} \tag{3.9}$$

where $i = 1, 2, \dots, N$.
(3) The reality of $\Omega$ and $W$ is equivalent to the relations $\Im h_{\pm}(\alpha_{2k}) = 0$ and hence to the relations $\Im h(\alpha_{2k}) = 0$ for all $k = 0, 1, \dots, N$.

## The MI Conditions and the Genus

The moment conditions (3.5) and the integral conditions (3.9) compose the system $F(\alpha, x, t) = F_N(\alpha, x, t) = 0$ alluded to earlier, from which the branch points $\alpha = \alpha(x, t)$ are calculated. We label this system of equations as the *MI* conditions. Solving the system of the MI conditions is a major part of this work. Another major part is the construction (or proof of existence) of a branch $\gamma^+$ of $\Im h(z) = 0$ that starts from $z = \frac{\mu}{2}$, passes through all these points, and connects to $-\frac{\mu}{2}$ in a way that the sign structure (2.29) is observed. Possibly, the MI conditions are solvable for more than one value of the genus. The correct genus is decided by the existence of the connection $\gamma^+$ described above.

**Behavior of $h(z)$ near the Branch Points; Degeneracy**

Expression (3.4) for $h'$ indicates that $h'(z)$ equals $(z-\alpha_{2k})^{1/2}$ $\times$ an analytic function when $z$ is near $\alpha_{2k}$; thus, near $\alpha_{2k}$, and for $\alpha$ that satisfy the MI conditions, we have

$$h_\pm(z) = (C_{2k})_\pm + (z-\alpha_{2k})^{3/2} \times \text{analytic function}, \quad \text{where } \Im(C_{2k})_\pm = 0\,. \tag{3.10}$$

Here $C_{2k\pm}$ are constants of integration that can be easily calculated in terms of $W$ and $\Omega$. Let $z = z_0$ be a zero of the integral in (3.4) that equals $h'(z_0)/R(z_0)$. If $z_0$ is on the contour $\gamma$, the following theorem allows us to include it and its complex conjugate in the chain of the $\alpha_j$'s. If $z_0$ is not a branch point, then it is introduced as a degenerate one with multiplicity 2. The *multiplicity* of $\alpha_j$ is understood as its multiplicity in the polynomial under the radical of $R$. If $z_0$ is a branch point, then including it raises the multiplicity of the branch point by 2. To maintain the symmetry we have to treat point $\bar{z}_0$ similarly.

THEOREM 3.1 (Degeneracy Theorem)

(1) *Suppose*

$$\frac{h'(z_0)}{R(z_0)} = 0 \quad \left(\textit{by symmetry, also } \frac{h'(\bar{z}_0)}{R(\bar{z}_0)} = 0\right) \tag{3.11}$$

*for some point $z_0 \in \gamma$. Then we have the following*:

(a) *Replacing $R(z)$ in (3.4) with $\tilde{R} = R(z)(z-z_0)(z-\bar{z}_0)$ (the multiplicities of $z_0$ and $\bar{z}_0$ are thus increased by 2) does not change the functions $h'(z)$ and $h(z)$, i.e., $h'(z;\tilde{R}) = h'(z;R)$ and $h(z;\tilde{R}) = h(z;R)$.*

(b) *If the original $\alpha$ satisfy the MI conditions with genus $2N$, then the new $\alpha$, corresponding to $\tilde{R}$, also satisfy the MI conditions with genus $2(N+1)$.*

(2) *Conversely, if a degenerate $\alpha = (\alpha_0, \alpha_2, \dots \alpha_{4N+1})$ with $\alpha_{2k} = \alpha_{2k+2} = z_0$ satisfies the MI conditions with genus $2N$, then the $\alpha$ that is obtained by removing the degenerate pair and its complex conjugate satisfies the MI conditions for genus $2(N-1)$. Furthermore, after the removal, $h'/R = 0$ at the site $z_0$ of the removed pair.*

PROOF: We simplify the notation by writing $z_0$ for $z_0$ in the proof. The proof of the first statement is based on the identity

$$\frac{1}{(\zeta - z)(\zeta - z_0)(\zeta - \bar{z}_0)} = \frac{c_1}{\zeta - z} + \frac{c_2}{\zeta - z_0} + \frac{c_3}{\zeta - \bar{z}_0}\,, \tag{3.12}$$

where

$$c_1 = \frac{1}{(z - z_0)(z - \bar{z}_0)}\,, \quad c_2 = \frac{1}{(z_0 - z)(z_0 - \bar{z}_0)}\,, \\ c_3 = \frac{-1}{(\bar{z}_0 - z)(z_0 - \bar{z}_0)}\,. \tag{3.13}$$

Inserting the expression for $\tilde{R}$ in formula (3.4) for $h'(z; \tilde{R})$ and utilizing the above identity, we obtain

$$\begin{aligned} 2\pi i h'(z; \tilde{R}) &= R(z) \int_{\hat{\gamma}} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)}\, d\zeta \\ &\quad - \frac{R(z)(z - \bar{z}_0)}{z_0 - \bar{z}_0} \int_{\hat{\gamma}} \frac{f'(\zeta)}{(\zeta - z_0)R(\zeta)}\, d\zeta \\ &\quad + \frac{R(z)(z - z_0)}{z_0 - \bar{z}_0} \int_{\hat{\gamma}} \frac{f'(\zeta)}{(\zeta - \bar{z}_0)R(\zeta)}\, d\zeta \\ &= 2\pi i h'(z; R)\,. \end{aligned} \tag{3.14}$$

The last equality holds because the second and third integrals equal $h'(z_0)/R(z_0)$ and $h'(\bar{z}_0)/R(\bar{z}_0)$, respectively, and vanish by hypothesis. The equality $h(z; \tilde{R}) = h(z; R)$ follows, since both satisfy (3.7).

The second statement on the MI conditions is obvious; $h$ does not change; thus, the behavior at infinity and the jumps remain the same.

In the third statement, removing the pair removes a zero from $R$; thus $h'/R = 0$ at the site of the removed pairs. The second and third integrals in (3.14) vanish. □

If the new $R = \tilde{R}$ still gives $h'(z_0)/R(z_0) = 0$, we may again redefine $R$ by multiplying by the factor $(z - z_0)(z - \bar{z}_0)$ one more time. We may proceed in this way until $h'(z_0)/R(z_0) \neq 0$ for all branch points.

Our strategy is to construct the solution of the system MI and the curve $\gamma$ at $t = 0$ and let the solution evolve.

THEOREM 3.2 (Evolution Theorem) *Let* $\alpha = (\alpha_0, \alpha_2, \alpha_4, \ldots, \alpha_{4N})$ *with distinct* $\alpha_{2k}$ *be a solution of* (2.30) *with genus* $2N$ *at some point* $(x_0, t_0)$. *Then*

- *the solution* $\alpha(x, t)$ *can be continued uniquely with the same genus into a neighborhood of* $(x_0, t_0)$*, and* $\alpha(x, t)$ *is a smooth function of* $x$ *and* $t$,
- $W$ *and* $\Omega$ *are smooth functions of* $x$ *and* $t$*, and*
- *if the function* $h(z) = h(z; \alpha(x, t), W(x, t), \Omega(x, t))$ *satisfies conditions* (2.29) *at* $(x_0, t_0)$*, then it also satisfies these conditions in a neighborhood of* $(x_0, t_0)$.

The proof is based on the implicit function theorem and the following expression for the Jacobian $\frac{\partial F}{\partial \alpha}$; see Section 6.1.

THEOREM 3.3 (Jacobian) *The Jacobian* $|\frac{\partial F}{\partial \alpha}|$ *is given by*

$$\left|\frac{\partial F}{\partial \alpha}\right| = \prod_{j=0}^{2N}\left|\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}\right|^2 \prod_{j<l}(\alpha_l-\alpha_j)\int_{\hat\gamma_{m,1}}\int_{\hat\gamma_{c,1}}\cdots\int_{\hat\gamma_{m,N}}\int_{\hat\gamma_{c,N}}\prod_{j<l}^{2N}(z_j-z_l)\prod_{k=1}^{2N}\frac{dz_k}{R(z_k)}, \tag{3.15}$$

*where* $h'(z)$ *is defined by* (3.4) *and the integral in* (3.15) *is equal to*

$$\det\begin{pmatrix} \int_{\hat\gamma_{m,1}}\frac{dz_1}{R(z_1)} & \int_{\hat\gamma_{c,1}}\frac{dz_2}{R(z_2)} & \int_{\hat\gamma_{m,2}}\frac{dz_3}{R(z_3)} & \cdots & \int_{\hat\gamma_{c,N}}\frac{dz_{2N}}{R(z_{2N})} \\ \int_{\hat\gamma_{m,1}}\frac{z_1dz_1}{R(z_1)} & \int_{\hat\gamma_{c,1}}\frac{z_2dz_2}{R(z_2)} & \int_{\hat\gamma_{m,2}}\frac{z_3dz_3}{R(z_3)} & \cdots & \int_{\hat\gamma_{c,N}}\frac{z_{2N}dz_{2N}}{R(z_{2N})} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{\hat\gamma_{m,1}}\frac{z_1^{2N-1}dz_1}{R(z_1)} & \int_{\hat\gamma_{c,1}}\frac{z_2^{2N-1}dz_2}{R(z_2)} & \int_{\hat\gamma_{m,2}}\frac{z_3^{2N-1}dz_3}{R(z_3)} & \cdots & \int_{\hat\gamma_{c,N}}\frac{z_{2N}^{2N-1}dz_{2N}}{R(z_{2N})} \end{pmatrix}. \tag{3.16}$$

According to Corollary 6.1, the Jacobian $|\frac{\partial F}{\partial \alpha}| \neq 0$ as long as all $\alpha_j$ are distinct and $\frac{h'(z)}{R(z)}|_{z-\alpha_j} \neq 0$, $j = 0, 1, \dots, 4N+1$. Then the evolution theorem implies that conditions (2.29) for the given $N$ hold in the process of any $(x, t)$ evolution as long as

- all $\alpha_j$ stay distinct,
- the ratio $h'(z)/R(z) \neq 0$ for any $z \in \gamma$, and
- the contour $\gamma$ stays away from singularities of $f(z)$ in $\mathbb{C}$.

*Breaking* occurs at some $(x, t)$ at which one of the above conditions is violated. Generically, it occurs on curves in the $(x, t)$–plane that we call *breaking curves*, across which there is a jump in the genus $2N$. A set of $\alpha$ satisfying the first two of the above conditions is called *nondegenerate*. Degeneracy can occur as the result of

- a collision between some $\alpha_j$ and a singular point of $f(z)$, including points on $\mathbb{R}$ and $\infty$ (note that in the latter two cases, $\alpha_j$ also collides with $\bar\alpha_j$),
- a collision between different $\alpha_{2k}$ in the upper half-plane (and the corresponding complex conjugates in the lower half-plane), and
- a collision between the contour $\gamma$, which is a zero level curve of $\Im h(z)$ (see (2.29)), (2.28), and any other branch of zero level curve of $\Im h(z)$.

Typically, a collision of some neighboring (along $\gamma$) $\alpha_{2k}$ in the upper half-plane leads to the decrease of the genus, whereas collision of $\gamma$ with another branch of the zero level curve of $\Im h(z)$ leads to the increase of the genus in accordance with the degeneracy theorem above. In a sense, these two events can be viewed as time reverses of each other and can be associated with the disappearance of a pair (or several pairs) of colliding $\alpha$'s and the appearance of a new pair (or several pairs) of $\alpha$'s at a point $z_0 \in \gamma$, such that $h'(z_0)/R(z_0) = 0$, i.e., at the point of collision of branches of $\Im h(z) = 0$.

In the case that we treat, the genus $2N = 2$, and we have $\alpha_0 \neq \alpha_2 = \alpha_4$. Note that the Jacobian $|\frac{\partial F}{\partial \alpha}|$ becomes zero if not all points in $\alpha$ are distinct. To establish the evolution through a breaking curve, we will use a reparametrization of the $\alpha$ that leads to a nonzero Jacobian.

When there are no degeneracies, the function $g(z)$ may be calculated directly through the formula

$$g(z) = \frac{R(z)}{2\pi i}\left(\int_{\gamma_m} \frac{f(\zeta)+W}{(\zeta - z)R_+(\zeta)}\,d\zeta + \int_{\gamma_c} \frac{\Omega}{(\zeta - z)R_+(\zeta)}\,d\zeta\right) \tag{3.17}$$

that expresses the unique $L_2^{\mathrm{loc}}$ solution of the first problem (3.1) that is analytic at infinity. Of course, this must agree with the definition (3.6) of $g(z)$ as the primitive of $g'(z)$. Expressing the integrals over the main and complementary arcs (3.17) in terms of the above loop integrals (see Figure 3.2), we obtain

$$\begin{aligned} g(z) = \frac{R(z)}{4\pi i}\Bigg[ &\oint_{\hat\gamma} \frac{f(\zeta)}{(\zeta - z)R(\zeta)}\,d\zeta \\ &+ \sum_{i=1}^{N}\left(\oint_{\hat\gamma_{m,i}} \frac{W_i}{(\zeta - z)R(\zeta)}\,d\zeta + \oint_{\hat\gamma_{c,i}} \frac{\Omega_i}{(\zeta - z)R(\zeta)}\right)d\zeta\Bigg], \end{aligned} \tag{3.18}$$

where the paths of integration $\hat\gamma$, $\hat\gamma_{m,i}$, and $\hat\gamma_{c,i}$ are contractible to their corresponding arcs without passing through $z$; the integrand is nonsingular for every such $z$. By deforming $\hat\gamma$ so that now $z$ is inside the loop $\hat\gamma$ and still outside the loops $\hat\gamma_{m,i}$ and $\hat\gamma_{c,i}$, we obtain

$$\begin{aligned} h(z) = \frac{R(z)}{2\pi i}\Bigg[ &\oint_{\hat\gamma} \frac{f(\zeta)}{(\zeta - z)R(\zeta)}\,d\zeta \\ &+ \sum_{i=1}^{N}\left(\oint_{\hat\gamma_{m,i}} \frac{W_i}{(\zeta - z)R(\zeta)}\,d\zeta + \oint_{\hat\gamma_{c,i}} \frac{\Omega_i}{(\zeta - z)R(\zeta)}\right)d\zeta\Bigg]. \end{aligned} \tag{3.19}$$

To obtain a workable expression for the first integral in these formulae, we deform the figure 8 contour $\hat\gamma$ as in Figure 6.3.

Note that the formula (3.18) for $g$ allows *degeneracies* in which main and complementary arcs collapse to points, leading to $\alpha$ with points that are not distinct. Indeed:

- If the complementary interval $\gamma_{c,k}^{+}$ collapses to a point, then (3.8) yields $W_k = W_{k-1}$, the common value factors out of two terms in (3.18), and the contours surrounding the two adjacent main arcs in the upper half-plane can be written as one contour surrounding both. No contour in the expression of $g(z)$ passes through the point of the collapsed interval, and the collapsing produces no singularity in the formula.

- Similarly, if a main arc $\gamma_{m,k}^{+}$ collapses to a point, then $\Omega_k = \Omega_{k+1}$, the common value factors from two terms in (3.18), and the contours corresponding to the two adjacent complementary arcs can be replaced by a single one. No contour crosses to the lower sheet at the point of the collapsed main arc, and the collapse occurs without the appearance of a singularity in (3.18).

# 4 Prebreak Evolution

In this section we will obtain the leading-order term of $q_0(x,t,\varepsilon)$ in the genus 0 region, i.e., in the region between the axis $t = 0$ and the breaking curve $t_0(x)$ of the $(x,t)$–plane; see Figure 1.2. We focus on finding the function $g(z)$ satisfying the conditions (2.19) for $N = 0$. In the case $N = 0$ the system (2.30) is solved for all $x \geq 0$, and solutions are found explicitly. That allows us to construct $g$ and $h$ in closed form. We then show that for a given $x$, the topology of the level curve $\Im h = 0$ is right (i.e., all conditions (2.19) are satisfied) for all $t \in [0, t_0(x))$.

## 4.1 Equation for the Branch Point $\alpha_0 = \alpha$

In the case $N = 0$, equations (2.30) consist of two moment conditions

$$g'(\infty) = 0 \quad \text{and} \quad zg'(z)\big|_{z=\infty} = 0 \tag{4.1}$$

to determine the branch point $\alpha = a + ib$. In the integral form, these conditions become

$$\int_{\gamma_m} \frac{f'(\zeta)}{R_+(\zeta)}\, d\zeta = 0\,, \qquad \int_{\gamma_m} \frac{\zeta f'(\zeta)}{R_+(\zeta)}\, d\zeta = 0\,, \tag{4.2}$$

where

$$\begin{aligned} f'(z,\varepsilon) &= -\frac{i\pi}{2} - \ln\left(\frac{\mu}{2} - z\right) + \frac{1}{2}\ln(z^2 - T^2) - x - 4tz\,, \\ zf'(z,\varepsilon) &= -\frac{i\pi}{2}z - z\ln\left(\frac{\mu}{2} - z\right) + \frac{z}{2}\ln(z^2 - T^2) - xz - 4tz^2\,. \end{aligned} \tag{4.3}$$

The aim of this subsection is to show that for any given $x \geq 0$ and $t > 0$, there exist two different solutions $\alpha_j(x,t) = a_j(x,t) + ib_j(x,t)$, $j = 1, 2$, satisfying (4.2). There is one special case $\alpha_1(0,t_0) = \alpha_2(0,t_0) = ib_0$, where $t_0 = \frac{1}{2(\mu+2)}$ and $b_0 = \sqrt{\mu+2}$. For a fixed $x$, the continuous curve $\alpha(x,t) = a(x,t) + ib(x,t)$ is referred to as a *trajectory* of $\alpha$ on $\mathbb{C}$, whereas for a fixed $t$ the continuous curve $\alpha(x,t)$ is called an *isochronic curve* on $\mathbb{C}$.

THEOREM 4.1

(i) *System* (4.2) *can be written as*

$$\begin{cases} \sqrt{(a-T)^2 + b^2} + \sqrt{(a+T)^2 + b^2} = \mu + 4tb^2 \\ \left[a - T + \sqrt{(a-T)^2 + b^2}\,\right]\left[a + T + \sqrt{(a+T)^2 + b^2}\,\right] = b^2 e^{2(x+4ta)}. \end{cases} \tag{4.4}$$

*In the particular case $\mu = 2$, the system* (4.4) *becomes*

$$\begin{cases} \sqrt{a^2+b^2} = 1 + 2tb^2 \\ a + \sqrt{a^2+b^2} = be^{(x+4ta)}. \end{cases} \tag{4.5}$$

(ii) *For any $x \geq 0$, $t > 0$, there exist two different solutions $\alpha_j(x,t)$, $j = 1,2$, to* (4.4) *such that $a_1(x,t) \geq 0$ and $a_2(x,t) \leq 0$. The only exception is the point $(0, t_0)$, where $\alpha_1(0,t_0) = \alpha_2(0,t_0) = ib_0$. Here $t_0 = \frac{1}{2(\mu+2)}$ and $b_0 = \sqrt{\mu+2}$. In the limit $t \to 0$, the point $\alpha_2 \to \infty$, whereas $\alpha_1(x,0) = \frac{\mu}{2}\tanh x + i \operatorname{sech} x$. The only cases $\alpha_{1,2}$ that are purely imaginary are the cases $x = 0$, $0 < t \leq t_0$.*

PROOF: The proof of part (i) is given in Appendix C. To prove (ii), we modify system (4.4) by introducing the auxiliary variables

$$u = 4ta + x\,, \quad \sinh p = \frac{a-T}{b}\,, \quad \sinh q = \frac{a+T}{b}\,. \tag{4.6}$$

Then, system (4.4) becomes

$$\begin{cases} p + q = 2u \\ \cosh p + \cosh q = \frac{\mu}{b} + 4tb, \end{cases} \tag{4.7}$$

which can be immediately transformed into

$$\begin{cases} p + q = 2u \\ \cosh \frac{1}{2}(p-q) = \frac{\mu+4tb^2}{2b\cosh u}. \end{cases} \tag{4.8}$$

Solving now (4.6) for $a$ and $T$, we get

$$a = \frac{1}{2}(\mu + 4tb^2)\tanh u \tag{4.9}$$

and

$$T = b\cosh u\sqrt{\cosh^2\frac{1}{2}(p-q) - 1} = \sqrt{a^2\coth^2 u - b^2\cosh^2 u}\,. \tag{4.10}$$

Thus, we get the system

$$\begin{cases} u = 4ta + x \\ a = \frac{1}{2}(\mu + 4tb^2)\tanh u \\ b^2 = \frac{a^2}{\sinh^2 u} - \frac{T^2}{\cosh^2 u}. \end{cases} \tag{4.11}$$

In the particular case $t = 0$, the system (4.11) yields $u = x$, $a = \frac{\mu}{2}\tanh x$, and $b = \operatorname{sech} x$, so that the time-zero isochronic curve is the arc of the ellipse connecting the points $\frac{\mu}{2}$ and $i$ in the first quadrant; see Figure 4.3.

Assume now that $a(x,t) = 0$. Then, according to (4.11), $x = u = 0$. Substituting $a = 0$ into the first equation in (4.4), we get the biquadratic equation

$$b^2 + T^2 = \left(\frac{\mu}{2} + 2tb^2\right)^2, \tag{4.12}$$

and, subsequently,

$$b^2 = \frac{1 - 2\mu t \pm \sqrt{(1-2\mu t)^2 - 16t^2}}{8t^2}\,. \tag{4.13}$$

At the moment $t = 0$, the negative branch of (4.13) yields $b_1 = 1$, whereas the positive branch yields $b_2 = \infty$. As $t$ increases from 0, the (positive) values of $b_{1,2}$ increase or decrease, respectively, according to (4.13), until $t$ reaches the critical value $t_0 = \frac{1}{2(\mu+2)}$. At this point $b_1 = b_2 = b_0 = \sqrt{\mu+2}$. For $t > t_0$, the discriminant in (4.13) becomes negative, so that our assumption $a(0,t) = 0$ becomes invalid. (It will be shown later that this is the break point for $x = 0$.)

Let us now assume that either $a < 0$ or $a > 0$. Then the variable $u \neq 0$ and has the same sign as $a$. If $a > 0$, then the first equation in (4.11) implies $u \geq x$. The corresponding requirement for $u < 0$ will be specified below. Such values of $u$ are called *admissible*. We want to show that any pair of $x \geq 0$ and of admissible $u$ uniquely determines $t$ and, thus, according to (4.11), uniquely determines $a$ and $b$.

Solving the first equation in (4.11) for $a$ and eliminating $b$ from the second and third equations, we obtain the quadratic equation for $t$

$$16T^2t^2\tanh^2 u - 4\mu t\sinh^2 u + (u-x)[\sinh 2u - (u-x)] = 0\,, \tag{4.14}$$

which has the solution

$$t = \frac{\frac{\mu}{2}\sinh 2u \pm \sqrt{(\frac{\mu}{2}\sinh 2u)^2 - 4T^2(u-x)[\sinh 2u - (u-x)]}}{8T^2\tanh u}\,. \tag{4.15}$$

Let us first show that the discriminant

$$D(u) = \Big(\frac{\mu}{2}\sinh 2u\Big)^2 - 4T^2(u-x)[\sinh 2u - (u-x)] \geq 0 \quad \text{for all } u \in \mathbb{R}.$$

Indeed, $D(x) > 0$. For any $u \neq x$ there exists some $k \in \mathbb{R}$ such that $\sinh 2u = k(u-x)$. Substituting this value into $D(u)$, we obtain after some algebra that

$$D(u) = \frac{\mu^2(u-x)^2}{4}\left[\Big(k - 2 + \frac{8}{\mu^2}\Big)^2 + \frac{16}{\mu^2}\Big(1 - \frac{4}{\mu^2}\Big)\right] > 0 \tag{4.16}$$

if $\mu > 2$. In the case $\mu < 2$ it is clear that $D(u) > 0$ if $u \geq x$ or $u \leq u_0$, where the value $u_0 < 0$ is determined by $\sinh 2u_0 = u_0 - x$; see Figure 4.1. In the case $\mu = 2$ equation (4.14) is linear in $t$.

Let us now show that conditions $t \geq 0$ and $b^2 \geq 0$ require the negative branch of (4.15) when $u \geq x$ and the positive branch when $u < 0$. If $\mu < 2$, this conclusion follows immediately from (4.15). So, let us consider $\mu > 2$. If $u < 0$, the condition $t \geq 0$ requires $u \leq u_0$. Indeed, the right-hand side of the third equation (4.11) is nonnegative. Substituting there $a = \frac{u-x}{4t}$, we obtain after some algebra $16T^2\xi^2 \leq (u-x)^2$, where $\xi = t\tanh u$. Combining this inequality with (4.14) yields

$$4\mu t\sinh^2 u \leq (u-x)\sinh 2u\,. \tag{4.17}$$

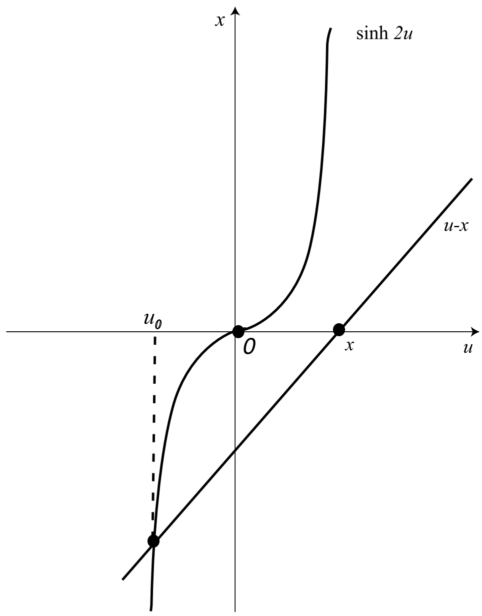


FIGURE 4.1. Determination of $u_0$.

In the case $u \geq x$ the latter inequality is reduced to

$$2\mu\xi \leq (u - x)\,. \tag{4.18}$$

Substituting $t$ from (4.15) into (4.18), we obtain

$$\pm\,\mu\sqrt{\left(\frac{\mu}{2}\sinh 2u\right)^2 - 4T^2(u-x)[\sinh 2u - (u-x)]} \leq -\frac{1}{2}\mu^2\sinh 2u + 4T^2(u-x)\,. \tag{4.19}$$

It is easy to see that the right-hand side of (4.19) is negative, so that the positive branch in (4.15) cannot be a solution. Inequality (4.19) with the negative square root reduces to the obvious

$$16(u-x)^2 \geq 0\,. \tag{4.20}$$

If $u \geq x$, it is also easy to see that $t \geq 0$.

In the case $u < 0$, inequality (4.17) becomes

$$2\mu\xi \geq (u - x)\,, \tag{4.21}$$

so that inequality (4.19) changes its sign to the opposite. It is now clear that the choice of the negative root in (4.19) will lead to the inequality opposite to (4.20), which is false. The choice of the positive root needs to be justified only in the case $-\frac{1}{2}\mu^2\sinh 2u + 4T^2(u-x) > 0$. But in this case we again arrive at the obvious (4.20). Thus, we need to choose the positive branch of (4.14) if $u < 0$. In this case,

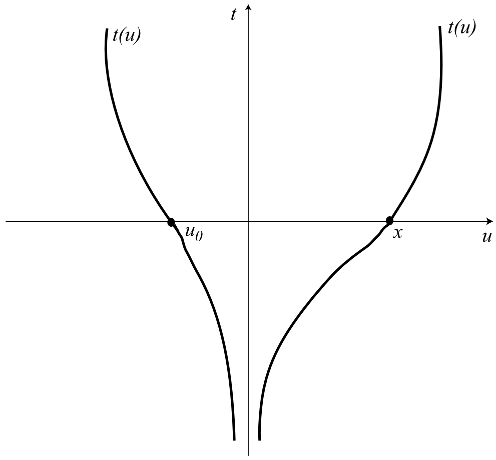


FIGURE 4.2. Graph $t(u)$.

the condition $t \geq 0$ leads to the requirement $u \leq u_0$. Therefore, any value of $u \geq x$ or $u \leq u_0$ is admissible. Thus, we have established the uniqueness of $t = t(u)$ for any fixed $x \geq 0$ and any admissible $u$.

To complete the proof, it only remains to show that for any given $x > 0$, $t > 0$, or $x = 0$, $t > t_0$, there exist exactly two corresponding values of $u$, one positive and one negative; see Figure 4.2. Equations (4.11) and (4.14) show immediately that $t(x) = t(u_0) = 0$ and $t \to \infty$ as $u \to \pm\infty$. Thus, it remains to show that $u\frac{dt}{du} > 0$ for any admissible $u$.

To this end, let us rewrite (4.14) as

$$16T^2\xi^2 - 2\mu\xi \sinh 2u + (u-x)[\sinh 2u - (u-x)] = 0\,, \tag{4.22}$$

where $\xi = t \tanh u$. Using implicit differentiation, we obtain

$$\frac{d\xi}{du} = \frac{4\mu\xi \cosh 2u - \sinh 2u - 2(u-x)[\cosh 2u - 1]}{32T^2\xi - 2\mu \sinh 2u}\,. \tag{4.23}$$

Taking into account $\frac{d\xi}{du} = \frac{dt}{du}\tanh u + \frac{2\xi}{\sinh 2u}$ and (4.22), we obtain after some algebra

$$4\sinh^2 u \frac{dt}{du} = \frac{4[2\mu\xi - (u-x)]\sinh^2 u \sinh 2u - [\sinh 2u - 2(u-x)]^2}{16T^2\xi - \mu \sinh 2u}\,. \tag{4.24}$$

Note that the numerator is always less than or equal to 0, according to (4.18) and (4.21), but the denominator is

$$\pm 2\sqrt{\left(\frac{\mu}{2}\sinh 2u\right)^2 - 4T^2(u-x)[\sinh 2u - (u-x)]}\,,$$

according to (4.15). Therefore, $u\frac{dt}{du} > 0$ is proven for all admissible $u$. The proof is completed. □

It is clear that at $t = 0$ we should choose $\alpha = \alpha_1$ to match our initial data (1.2); see Section 4.4 for details. Thus, we must choose $\alpha = \alpha_1$ for the entire genus 0 region.

Note that for a fixed $x \geq 0$ the positive value of $u$ monotonically increases with $t$, whereas the negative value of $u$ is monotonically decreasing (if $x = 0$ it is required that $t \geq t_0$). In a sense, $u$ can be considered as a new "time" for the fixed $x$. Given a pair $(x, t)$ (we assume $t > t_0$ if $x = 0$), the corresponding values of $(a, b)$ can be considered, according to (4.11), as the intersection of the two hyperbolas in the $(a, b)$–plane given by

$$\begin{cases} 2a^2 \coth u - \mu a - (u - x)b^2 = 0 \\ \frac{a^2}{\sinh^2 u} - b^2 = \frac{T^2}{\cosh^2 u}. \end{cases} \tag{4.25}$$

These equations yield

$$[\sinh 2u - (u - x)]a^2 - \mu a \sinh^2 u + (u - x) \tanh^2 u T^2 = 0\,, \tag{4.26}$$

so that

$$\begin{aligned} a &= \frac{\mu \sinh^2 u + Q}{2[\sinh 2u - (u - x)]} \\ &= \frac{\mu \sinh^2 u + \sqrt{(\mu \sinh^2 u)^2 - 4T^2(u - x)[\sinh 2u - (u - x)] \tanh^2 u}}{2[\sinh 2u - (u - x)]} \end{aligned} \tag{4.27}$$

and

$$b = \frac{\sqrt{2(1 - T^2) \sinh^2 u + 2T^2(u - x) \tanh u + \frac{\mu}{2} Q}}{\sinh 2u - (u - x)} \tag{4.28}$$

are the positive solutions to (4.4) (the corresponding $u$ is positive). Equations (4.27)–(4.28) give explicit formulae for $\alpha_1(x, t)$ in terms of $x$ and $u$. The expressions for $\alpha_2(x, t)$ require the choice of negative $u$ and different branches of the square roots.

In the particular case $T = 0$ (i.e., $\mu = 2$), expressions

$$\begin{gathered} a = \frac{2 \sinh^2 u}{\sinh 2u - (u - x)}\,, \quad b = \frac{2 \sinh u}{\sinh 2u - (u - x)}\,, \\ t = \frac{(u - x)[\sinh 2u - (u - x)]}{8 \sinh^2 u}\,, \end{gathered} \tag{4.29}$$

for $a$, $b$, and $t$ in terms of $x$ and $u$ follow from (4.14), (4.25), and (4.26). In the particular case $\mu = 0$, the corresponding expressions are

$$a^2 = \frac{(u-x)\tanh^2 u}{\sinh 2u - (u-x)}\,, \quad b^2 = \frac{2\tanh u}{\sinh 2u - (u-x)}\,, \quad t = \frac{1}{4}\sqrt{(u-x)[\sinh 2u - (u-x)]}\coth u\,. \tag{4.30}$$

COROLLARY 4.2 *The limit* $\lim_{t\to\infty} \alpha_1(x,t) = \frac{\mu}{2}$. *The function* $\alpha_1(x,t)$ *yields a one-to-one correspondence between the first quadrant* $x \geq 0$, $t \geq 0$, *and the region* $U$ *bounded between the curves* $\alpha_1(x,0)$ *and* $\alpha_1(0,t)$; *the curves are included. The corresponding result is also true for* $\alpha_2(x,t)$.

PROOF: Equation (4.14) shows that $t \to \infty$ implies $u \to \infty$. Then the asymptotic of $\alpha_1(x,t)$ follows from (4.25) and (4.26). For any $z \in U$, the corresponding values of $x$ and $t$ are uniquely determined by (4.4). The easiest way to show the one-to-one correspondence between $U$ and $x, t \geq 0$, is to consider the differential equations $\alpha_x = -F_\alpha^{-1}F_x$ and $\alpha_t = -F_\alpha^{-1}F_t$, obtained from equation (2.30): $F(\alpha, x, t) = 0$. As will be shown in Section 6.3, these differential equations are autonomous (in fact, their right-hand sides depend only on $\alpha$), and the only singularity they have in $U$ is $i\sqrt{\mu+2}$. Since

- the isochronic curve $\alpha(x,0)$ contains the set of initial values for all $x \geq 0$ for the equation $\alpha_x = -F_\alpha^{-1}F_x$ and trajectories directed into $U$,
- trajectories $\alpha(x,t)$ with different $x$ do not intersect, and
- $b(x,t) > 0$ for all finite nonnegative $x$ and $t$,

hence all trajectories $\alpha(x,t)$ lie in $U$; see Figure 4.3. Thus, according to the theorem, for any pair of nonnegative $(x,t)$ there exists $z \in U$ such that $\alpha_1(x,t) = z$. The proof is completed. □

Notice that the system (4.11) is invariant under the transformation $x \mapsto -x$, $a \mapsto -a$, $b \mapsto b$, and $u \mapsto -u$.

## 4.2 Calculation of *g*

In the case of $N = 0$, the first RHP (3.1) can be written as

$$g_+ + g_- = f\,, \quad \Im(g_+ - g_-) = 0\,, \quad \text{on } \gamma_m\,, \tag{4.31}$$

where $\gamma_m$ is an unknown, simple, oriented, and symmetrical contour passing through the points $\bar\alpha$, $\frac{\mu}{2}$, and $\alpha$, and $g$ is a function, analytic everywhere in $\bar{\mathbb{C}} \setminus \gamma_m$. The analyticity of $f$ implies that the analytic function $g(z)$ depends only on the endpoints of $\gamma_m$, but not on $\gamma_m$ itself.

Let $R(z) = \sqrt{(z-\alpha)(z-\overline{\alpha})}$, and denote the Cauchy integral operator

$$C_{\gamma_m}[f](z) = \frac{1}{2\pi i}\int_{\gamma_m} \frac{f(\zeta)}{(\zeta - z)}\,d\zeta\,, \quad z \notin \gamma_m\,. \tag{4.32}$$

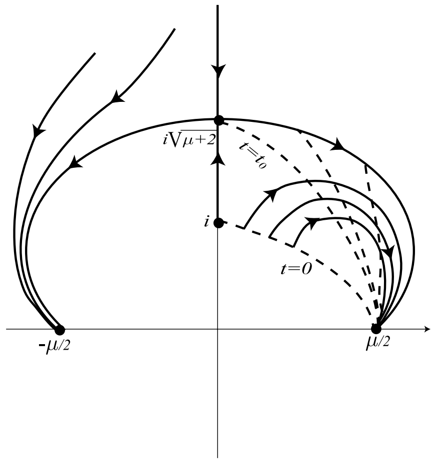


FIGURE 4.3. The trajectories and isochrones of $\alpha_1(x,t)$, $t_0 = \frac{1}{2}(\mu+2)$.

Our choice of the branch of $R(z)$ is such that $R(z) \to -z$ as $z \to \infty$.

For $z \in \gamma_m$, let $C^{\pm}_{\gamma_m}$ define the positive (negative) limit of $C_{\gamma_m}[f](\zeta)$ as $\zeta \to z$ from the positive (negative) sides. It is well-known that

$$C^{+}_{\gamma_m} - C^{-}_{\gamma_m} = \mathrm{Id}$$

and

$$\left[C^{+}_{\gamma_m} + C^{-}_{\gamma_m}\right][f](z) = \frac{1}{\pi i}\int_{\gamma_m} \frac{f(\zeta)}{\zeta - z}\,d\zeta\,, \qquad z \in \gamma_m\,.$$

It is also well-known that

$$g_{\pm}(z) = R_{\pm}(z) C^{\pm}_{\gamma_m}\left[f R_{+}^{-1}\right] \tag{4.33}$$

solves the RHP (4.31) for $g$, where $R_{\pm}$ denotes the limiting values of $R$ from the positive (negative) sides. Indeed,

$$\begin{aligned} g_{+} + g_{-} &= R_{+} C^{+}_{\gamma_m}\left[f R_{+}^{-1}\right] + R_{-} C^{-}_{\gamma_m}\left[f R_{+}^{-1}\right] \\ &= R_{+}\left(C^{+}_{\gamma_m} - C^{-}_{\gamma_m}\right)\left[f R_{+}^{-1}\right] = R_{+} f R_{+}^{-1} = f\,. \end{aligned}$$

Since the RHP (4.31) for $g$ is additive, we can look for $g$ as a sum of individual solutions of the RHP for each additive term of (2.10). Direct calculations, based on (4.33) and the technique of Appendix C, yield

$$\begin{aligned} \frac{i\pi}{2}\left(\frac{\mu}{2} - z\right) \longmapsto\ & \frac{1}{2} R\left[\ln\left(\frac{\mu}{2} - a + \sqrt{\left(\frac{\mu}{2} - a\right)^2 + b^2}\right) - \ln b\right] \\ & - \frac{1}{2}\left(\frac{\mu}{2} - z\right)\left[\ln \frac{-\sqrt{(\frac{\mu}{2} - a)^2 + b^2}\,R + (z-a)(\frac{\mu}{2} - a) + b^2}{z - \frac{\mu}{2}} - \ln b\right]. \end{aligned} \tag{4.34}$$

In order to compute $g$ for the terms of (2.10) that contain logarithms, we need to solve the RHP (4.31) where $f(z) = (z + A)\ln(z + A)$ for some $A \in \mathbb{R}$. Then, using (4.33), we get

$$\begin{aligned} g(z) &= \frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{(\zeta + A)\ln(\zeta + A)}{(\zeta - z)R(\zeta)}\, d\zeta \\ &= \frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{\ln(\zeta + A)}{R(\zeta)}\, d\zeta + (z + A)\frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{\ln(\zeta + A)}{(\zeta - z)R(\zeta)}\, d\zeta\,. \end{aligned} \tag{4.35}$$

Utilizing (C.1) and one of the integrals from (C.6) from Appendix C, we obtain

$$\begin{aligned} \int_{\gamma_m} \frac{\ln(\zeta + A)}{R(\zeta)}\, d\zeta &= i \int_0^b \frac{\ln[(a + A)^2 + b^2]}{\sqrt{b^2 - \beta^2}}\, d\beta \\ &= i\pi \ln \frac{a + A + \sqrt{(a + A)^2 + b^2}}{2}\,. \end{aligned} \tag{4.36}$$

Utilizing the residue theorem, the second integral in (4.35) becomes

$$\frac{R(z)}{2\pi i} \int_{\gamma_m} \frac{\ln(\zeta + A)}{(\zeta - z)R(\zeta)}\, d\zeta = \frac{1}{2}\ln(z + A) - \frac{1}{2}R(z)\int_{-\infty}^{-A} \frac{d\zeta}{(\zeta - z)R(\zeta)}\,, \tag{4.37}$$

where the evaluation of the latter integral yields

$$\begin{aligned} &R^{-1}(z)\big[\ln(z + A) + \ln[R(z) - (z - a)] \\ &\qquad - \ln[\sqrt{(a + A)^2 + b^2}R(z) - (a + A)(z - a) + b^2]\big]\,. \end{aligned} \tag{4.38}$$

Thus, we obtain

$$\begin{aligned} g(z) &= \frac{1}{2}R(z) \ln \frac{a + A + \sqrt{(a + A)^2 + b^2}}{2} \\ &\quad + \frac{1}{2}(z + A) \ln \frac{\sqrt{(a + A)^2 + b^2}R(z) - (a + A)(z - a) + b^2}{R(z) - (z - a)}\,. \end{aligned} \tag{4.39}$$

Applying (4.39) to $(\frac{\mu}{2}-z)\ln(\frac{\mu}{2}-z)$ and $\frac{1}{2}[(z+T)\ln(z+T)+(z-T)\ln(z-T)]$ and taking into account (4.34), we obtain

$$
\begin{aligned}
g(z) = {} & \frac{\frac{\mu}{2}-z}{2}\left[\ln\frac{z-\frac{\mu}{2}}{-R(z)+z-a}+\ln b\right] \\
& + \frac{z+T}{4}\ln\left[\sqrt{(a+T)^2+b^2}R(z)-(a+T)(z-a)+b^2\right] \\
& + \frac{z-T}{4}\ln\left[\sqrt{(a-T)^2+b^2}R(z)-(a-T)(z-a)+b^2\right] \\
& - \frac{z}{2}\ln[R(z)-(z-a)]-t(z-a)R(z) \\
& - \frac{1}{2}T\tanh^{-1}\frac{2T}{\mu}-\frac{xz}{2}-tz^2+\frac{1}{4}(\mu\ln 2+\varepsilon\pi)\,.
\end{aligned} \tag{4.40}
$$

Subsequently, for $h=2g-f$, we obtain

$$
\begin{aligned}
h(z) = {} & \left(\frac{\mu}{2}-z\right)\left[\ln b-\frac{i\pi}{2}\right]-\frac{\mu}{2}\ln[R(z)-(z-a)]-2t(z-a)R(z) \\
& + \frac{z+T}{2}\ln\frac{\sqrt{(a+T)^2+b^2}\,R(z)-(a+T)(z-a)+b^2}{z+T} \\
& + \frac{z-T}{2}\ln\frac{\sqrt{(a-T)^2+b^2}\,R(z)-(a-T)(z-a)+b^2}{z-T}\,.
\end{aligned} \tag{4.41}
$$

In the particular case $\mu=2$, (4.40) and (4.41) become

$$
\begin{aligned}
g(z) = {} & \frac{1-z}{2}[\ln(1-z)+\ln b]+\frac{z}{2}\ln\left[\sqrt{a^2+b^2}R(z)-a(z-a)+b^2\right] \\
& - \frac{1}{2}\ln[R(z)-(z-a)]-t(z-a)R(z) \\
& - \frac{xz}{2}-tz^2+\frac{1}{4}(\mu\ln 2+\varepsilon\pi)\,, \\
h(z) = {} & (1-z)\left[\ln b-\frac{i\pi}{2}\right]+z\ln\frac{\sqrt{a^2+b^2}R(z)-a(z-a)+b^2}{z} \\
& - \ln[R(z)-(z-a)]-2t(z-a)R(z)\,.
\end{aligned} \tag{4.42}
$$

## 4.3 Functions $g_x$ and $g_t$

Functions $g_x$ and $g_t$ and related functions $h_x$ and $h_t$, studied here, play an important role in our further analysis.

LEMMA 4.3 *Expressions for $g_x$ and $g_t$, $x,t\geq 0$, are given by*

$$
g_x(z)=-\frac{1}{2}[z+R(z)] \quad \textit{and} \quad g_t(z)=-(z+a)R(z)-z^2\,. \tag{4.43}
$$

PROOF: Since $g(z) \sim O(z-z_0)^{3/2}$ when $z_0 = \alpha$ and $z_0 = \bar{\alpha}$, we can look for $g_x$ and $g_t$ as solutions of the RHPs

$$y_+ + y_- = f_x \quad \text{and} \quad y_+ + y_- = f_t \tag{4.44}$$

on $z \in \gamma$, respectively, where $f_x = -z$ and $f_t = -2z^2$. These RHPs have unique solutions in $L_2(\gamma_m)$, and it is easy to check that (4.43) are such solutions. □

COROLLARY 4.4 *As a result of Lemma* 4.3, *we obtain alternative expressions for* $g(z,x,t)$ *by*

$$\begin{aligned} g(z,x,t) &= g(z,0,t) - \frac{1}{2}\int_0^x [z + R(z,s,t)]ds\,, \\ g(z,x,t) &= g(z,x,0) - \int_0^t [(z+a)R(z,x,\tau) - z^2]d\tau\,, \end{aligned} \tag{4.45}$$

*where* $R(z,u,v) = -\sqrt{(z-a(u,v))^2 + b^2(u,v)}$.

Another immediate consequence of Lemma 4.3 and relations (2.19) and (2.20) is

$$h_x(z,x,t) = -R(z) \quad \text{and} \quad h_t(z,x,t) = -2(z+a)R(z)\,. \tag{4.46}$$

We now focus attention on the signs of $\Im h_x$ and $\Im h_t$ in the upper half-plane. The real axis $\mathbb{R}$ is a zero level curve for both $\Im h_x$ and $\Im h_t$. Additionally, the segment $[\bar{\alpha},\alpha]$ is a zero level curve for $\Im h_x$. Since $h_t(z) \sim O(z-\alpha)^{1/2}$ near $z = \alpha$ and $h_t \sim O(z^2)$ as $z \to \infty$, there also exists an additional zero level curve $\kappa$ that connects $\alpha$ and $i\infty$; see Figure 5.2. An equation for $\kappa$ will be derived in Lemma 5.3. It shows that $\kappa$ can either lie in the upper half-plane or have a bounded part lying in the lower half-plane. In the latter case, we replace this part of $\kappa$ by the corresponding segment of the real axis, so that $\Im z \geq 0$ for all $z \in \kappa$.

LEMMA 4.5 *For any* $x, t \geq 0$ *we have the following*:

(i) *The inequality* $\Im h_x(z) > 0$ *holds for all sufficiently large* $|z|$ *in the upper half-plane* $\Im z > 0$. *This inequality changes sign each time* $z$ *crosses either the contour* $\gamma_m^+$ *or the vertical segment* $[a,\alpha]$.

(ii) *The inequality* $\Im h_t(z) < 0$ *holds for all sufficiently large* $|z|$ *to the left of* $\kappa$. *It changes its sign each time* $z$ *crosses either through* $\kappa$ *or through* $\gamma_m^+$.

PROOF: (i) The zero level curve of $\Im h_x(z) = \sqrt{(z-\alpha)(z-\bar{\alpha})}$ is $\mathbb{R} \cup [\bar{\alpha}]$, where $[\bar{\alpha},\alpha]$ is the branch cut of the function $h_x(z)$; see Figure 4.4. On the contour $\gamma_m$, according to (2.21), the function $h_x$ satisfies

$$(h_x)_+ + (h_x)_- = 0\,. \tag{4.47}$$

Thus, $\Im h_x$ changes its sign each time $z$ crosses either $\gamma_m^+$ or $[a,\alpha]$. It is also easy to see that $\arg(-h_x(z)) = \frac{1}{2}[\arg(z-\alpha) + \arg(z-\bar{\alpha})] \in (0,\pi)$ if $\Im z > 0$ and $-\Re z$ is a sufficiently large positive number. For such $z$, $\Im h_x(z) > 0$. Part (i) is proven. The proof of part (ii) is similar to that for part (i). □

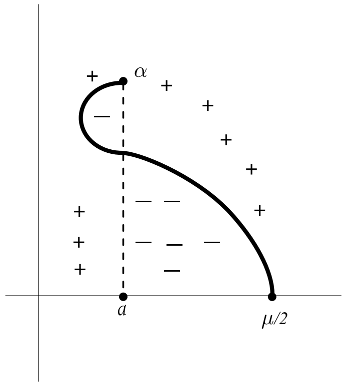


FIGURE 4.4. Zero level curves of $\Im h_x(z)$.

## 4.4 Leading-Order Prebreak Solution

If $m^{(0)}(z,x,t) = I + m_1^{(0)}/z + O(z^{-2})$ is the expansion of the solution $m$ to the RHP (2.1) near $z = \infty$, then the potential $q(x,t,\varepsilon)$ is given by

$$q = -2\big(m_1^{(0)}\big)_{12}\,; \tag{4.48}$$

see Section 2.1. As mentioned earlier, we now consider $m$ to be a solution to the RHP (2.11)–(2.12) instead of (2.1)–(2.2). In this section we compute the leading-order term $q_0(x,t,\varepsilon)$ of the potential $q(x,t,\varepsilon)$, corresponding to (2.11)–(2.12), and show that $q_0(x,0,\varepsilon)$ coincides with (1.2) as $\varepsilon \to 0$. This is done under the assumption that conditions (2.19) with $N = 0$ are satisfied in a region of the $(x,t)$–plane, called the *genus 0 region*, that contains the semi-axis $t = 0$, $x \geq 0$.

THEOREM 4.6 *In the zero genus region, the leading-order term* $q_0(x,t,\varepsilon)$ *corresponding to the RHP* (2.11)–(2.12) *is given by*

$$q_0(x,t,\varepsilon) = A(x,t)e^{\frac{i}{\varepsilon}S(x,t)}\,, \tag{4.49}$$

*where* $A(x,t) = b(x,t)$ *and* $S(x,t) = -2\int_0^x a(s,t)ds$*, and expressions for* $a$ *and* $b$ *are given in* (4.27)–(4.30).

PROOF: Combining (4.48) and (2.13), we obtain

$$q_0(x,t,\varepsilon) = -\big(2m_1^{(3)}\big)_{12}e^{4\frac{i}{\varepsilon}g(\infty)}\,. \tag{4.50}$$

According to (2.19) and (2.20), the diagonal entries of the jump matrix $V^{(3)}$, given by (2.15), are

$$e^{\pm 2\frac{i}{\varepsilon}(g_+ - g_-)} = e^{\pm 2\frac{i}{\varepsilon}h_+}$$

where $h_+ \in \mathbb{R}$ when $z \in \gamma$. Since $N = 0$, the nonzero off-diagonal element of $V^{(3)}$ is $-1$. Using the factorization

$$\begin{pmatrix} e^{2\frac{i}{\varepsilon}h_+} & 0 \\ -1 & e^{-2\frac{i}{\varepsilon}h_+} \end{pmatrix} = \begin{pmatrix} 1 & -e^{2i\frac{i}{\varepsilon}h_+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -e^{-2\frac{i}{\varepsilon}h_+} \\ 0 & 1 \end{pmatrix}, \tag{4.51}$$

which is a particular case of (2.16), we can reduce the RHP $P^{(3)}$ to the RHP $P^{(\mathrm{mod})}$:

$$m_+^{(\mathrm{mod})} = m_-^{(\mathrm{mod})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.52}$$

as discussed in Section 2.7. Using now the factorization

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \tag{4.53}$$

we obtain the solution to the RHP (4.52) as

$$m^{(\mathrm{mod})} = \frac{1}{2}\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}\left(\frac{z-\alpha}{z-\bar\alpha}\right)^{\frac{1}{4}\sigma_3}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}. \tag{4.54}$$

Then $(m^{(\mathrm{mod})})_{12} = -\frac{i}{2}(\beta - \beta^{-1})$, where

$$\beta = \left(\frac{z-\alpha}{z-\bar\alpha}\right)^{\frac{1}{4}} = 1 - \frac{ib}{2z} + O(z^{-2}).$$

Then $(m_1^{(\mathrm{mod})})_{12} = -\frac{1}{2}b$, so that

$$q_0(x,t,\varepsilon) = b(x,t)e^{4\frac{i}{\varepsilon}g(\infty)}, \tag{4.55}$$

as long as the conditions (2.19) are satisfied with $N = 0$. To complete the theorem, it is sufficient to mention that, according to Lemma 4.3, $g_x(\infty) = -\frac{1}{2}a$. □

In order to provide the expression for $S(x,t)$ in the closed form, let us now, according to (3.17) and (C.2), compute $g(\infty)$ by

$$g(\infty) = \frac{1}{\pi}\left[\int_0^b \Re f(a+i\beta)\frac{d\beta}{\sqrt{b^2-\beta^2}} - \int_a^{\frac{\mu}{2}} \frac{\Im[f(x)]dx}{\sqrt{b^2+(x-a)^2}}\right]. \tag{4.56}$$

Using (4.9) and integration formulae (C.6), Section C, we compute the first integral in (4.56) to be

$$\begin{aligned}
&-\frac{1}{2}\Bigg(T\tanh^{-1}\frac{T}{\frac{\mu}{2}} - b + \sqrt{b^2+\left(\frac{\mu}{2}-a\right)^2} - \left(\frac{\mu}{2}-a\right) \\
&\qquad + \frac{1}{2}\left[\sqrt{b^2+(a+T)^2} + \sqrt{b^2-(a-T)^2} - (a+T)\right]\Bigg) \\
&\quad + \frac{1}{2}\left(tb^2 + \left(\frac{\mu}{2}-a\right)\ln\frac{\frac{\mu}{2}-a+\sqrt{b^2+(\frac{\mu}{2}-a)^2}}{2}\right) + \frac{1}{4}(\mu\ln 2 + \varepsilon\pi) \\
&\quad + \frac{1}{4}\Bigg((a+T)\ln\frac{a+T+\sqrt{b^2+(a+T)^2}}{2} \\
&\qquad\qquad + (a-T)\ln\frac{a-T+\sqrt{b^2+(a-T)^2}}{2}\Bigg).
\end{aligned}$$

The corresponding computation of the second integral yields

$$\frac{1}{2}\left[\sqrt{b^2+\xi^2}-\left(\frac{\mu}{2}-a\right)\ln\left(\xi+\sqrt{b^2+\xi^2}\right)\right]\Big|_0^{\frac{\mu}{2}-a}$$
$$=\frac{1}{2}\Bigg(\sqrt{b^2+\left(\frac{\mu}{2}-a\right)^2}$$
$$-\left(\frac{\mu}{2}-a\right)\ln\left[\frac{\mu}{2}-a+\sqrt{b^2+\left(\frac{\mu}{2}-a\right)^2}\right]-b+\left(\frac{\mu}{2}-a\right)\ln b\Bigg).$$

Adding the latter two expressions and taking into account (4.4), we obtain

$$g(\infty)=\frac{1}{2}\Bigg(\frac{\mu}{2}\ln b+t(2a^2-b^2)$$
$$-T\left[\tanh^{-1}\frac{T}{\mu/2}-\frac{1}{2}\ln\frac{a+T+\sqrt{b^2+(a+T)^2}}{a-T+\sqrt{b^2+(a-T)^2}}\right]+\frac{1}{2}\varepsilon\pi\Bigg). \tag{4.57}$$

Equations (4.4) yield

$$\pm\sqrt{b^2+(a\pm T)^2}=\frac{\mu}{2}+2tb^2\pm\frac{2aT}{\frac{\mu}{2}+2tb^2},$$

so that the logarithmic term in (4.57) becomes $\tanh^{-1}\frac{T}{\mu/2+2tb^2}$. Thus, we get

$$g(\infty)=\frac{1}{2}\left[\frac{\mu}{2}\ln b+t(2a^2-b^2)-T\tanh^{-1}\frac{2Ttb^2}{T^2+\mu tb^2}+\frac{1}{2}\varepsilon\pi\right]. \tag{4.58}$$

Combined with (4.55), this equation yields the following corollary:

COROLLARY 4.7 *The leading-order term $q_0(x,t,\varepsilon)$, obtained through the RHP* (2.11)–(2.12) *according to* (4.50)*, can be written as*

$$q_0(x,t,\varepsilon)=$$
$$-b(x,t)\exp\left(2\frac{i}{\varepsilon}\left[\frac{\mu}{2}\ln b+t(2a^2-b^2)-T\tanh^{-1}\frac{2Ttb^2}{T^2+\mu tb^2}\right]\right). \tag{4.59}$$

The required accuracy follows from the results of Section 9.

COROLLARY 4.8 *The expression* (4.59) *with $t=0$ coincides with* (1.2)*.*

PROOF: Indeed, in this case (4.58) yields $g(\infty)=\frac{1}{4}(\mu\ln b+\varepsilon\pi)$. So, according to (4.55),

$$q_0=be^{\frac{i\mu}{\varepsilon}\ln b+i\pi}=-[\cosh(x)]^{-\frac{i\mu}{\varepsilon}-1}, \tag{4.60}$$

since, according to (4.11), $b(x,0)=\frac{1}{\cosh x}$. This answer coincides with our initial potential (1.2):

$$q=-\frac{1}{\cosh x}e^{-\frac{i\mu}{\varepsilon}\ln\cosh x}.$$

□

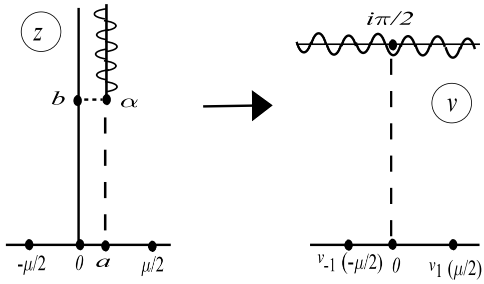


FIGURE 4.5. Hyperbolic variables.

In the particular cases $\mu = 2$ and $\mu = 0$, according to (4.29) and (4.30), we obtain

$$q_0(x,u) = -\left[\frac{2\sinh u}{\sinh 2u - (u-x)}\right]^{\frac{2i}{\varepsilon}+1} \exp\left(\frac{i}{\varepsilon}\frac{2\sinh^2 u - 1}{\sinh 2u - (u-x)}\right) \tag{4.61a}$$

and

$$\begin{aligned} q_0(x,u) = {} & -\sqrt{\frac{2\tanh u}{\sinh 2u - (u-x)}} \\ & \times \exp\left(\frac{i}{\varepsilon}\left[[(u-x)\tanh u - 1]\sqrt{\frac{u-x}{\sinh 2u - (u-x)}} \right.\right. \\ & \left.\left. - 2\tan^{-1}\sqrt{\frac{(u-x)\tanh u}{\sinh 2u - (u-x)}}\right]\right), \end{aligned} \tag{4.61b}$$

respectively, where $u \geq x \geq 0$.

## 4.5 Existence of the Contour $\gamma$ When $x = 0$

The results of Section 4.4 are based on the assumption that the function $g(z)$, constructed in Section 4.2, and the yet-to-be-determined contour $\gamma$ satisfy conditions (2.19). In this and in the following section we will establish the existence of such $\gamma$ for $x = 0$ and $x > 0$, respectively, and for some $t \geq 0$.

We start by introducing a new variable $v$ by

$$\sinh v = \frac{z-a}{b}, \tag{4.62}$$

which maps the upper $z$ half-plane with the cut $\Im z \geq b$, $\Re z = a$, into the strip $S = \{v : 0 \leq \Im v \leq \frac{\pi}{2}\}$; see Figure 4.5. Moreover, $\alpha \mapsto \frac{i\pi}{2}$, $T \mapsto -p$, $-T \mapsto -q$, and $\pm\frac{\mu}{2} \mapsto v_{\pm 1}$, where $p$ and $q$ are defined by (4.6), and $\pm v_{\pm 1}$ are positive numbers.

In the variables $v$, $p$, and $q$, we get $R(z) = b\cosh v$ and

$$\begin{gathered}\sqrt{(a+T)^2+b^2} = b\cosh q\,, \quad \sqrt{(a-T)^2+b^2} = b\cosh p\,,\\ z = b\left[\sinh v + \frac{1}{2}(\sinh p + \sinh q)\right],\end{gathered} \tag{4.63}$$

so that, using (4.7), we can rewrite (4.41) after some algebra as

$$\begin{aligned} h(v,p,q) = {} & \frac{1}{4}[\mu - b(\cosh q + \cosh p)]\sinh 2v \\ & + \frac{b}{2}(\sinh v + \sinh q)\ln i\frac{\cosh\frac{v-q}{2}}{\sinh\frac{v+q}{2}} \\ & + \frac{b}{2}(\sinh v + \sinh p)\ln i\frac{\cosh\frac{v-p}{2}}{\sinh\frac{v+p}{2}} + \frac{\mu}{2}\left(v - \frac{i\pi}{2}\right). \end{aligned} \tag{4.64}$$

Direct calculations show that $h(\frac{i\pi}{2}, p, q) = 0$ for all $p$ and $q$. In the particular case $\mu = 2$, we have $p = q = u$, so that (4.64) becomes

$$\begin{aligned} h(v,u) = {} & \frac{1}{2}[1 - b\cosh u]\sinh 2v \\ & + b(\sinh v + \sinh u)\ln i\frac{\cosh\frac{v-u}{2}}{\sinh\frac{v+u}{2}} + \left(v - \frac{i\pi}{2}\right). \end{aligned} \tag{4.65}$$

In the case $\mu = 0$ the first term of (4.64) becomes $-tb^2\sinh 2v$.

THEOREM 4.9 *The vertical interval $x = 0$, $0 \le t < \frac{1}{2(\mu+2)}$ of the $(x,t)$–plane belongs to the genus 0 region.*

PROOF: In order to prove existence of the contour $\gamma^+$ satisfying conditions (2.19) with $N = 0$, we study the level curves of $B$, where $h = A + iB$ in the strip $S$. In the case $\mu < 2$, i.e., $T$ is purely imaginary, we consider $S$ with the cut from the origin to $-p$.

Let us study $B$ on the real axis. If $p, q \in \mathbb{R}$, i.e., if $\mu \ge 2$, we get

$$B = \frac{\pi}{2}\left(b\sinh v + \frac{b}{2}(\sinh q + \sinh p) - \frac{\mu}{2}\right) = \frac{\pi}{2}\left(z - \frac{\mu}{2}\right) \tag{4.66}$$

when $v \ge -p$. Thus, $B > 0$ or $B < 0$ when $v > v_{+1}$ or $v < v_{+1}$, respectively. The corresponding expressions on the intervals $(-q, -p)$ and $(-\infty, -q)$ are $\frac{\pi}{2}(T - \frac{\mu}{2}) < 0$ and $-\frac{\pi}{2}(z + \frac{\mu}{2})$, respectively. Thus, $B$ is negative between $v_{\pm 1}$ and positive outside these values.

In the case $0 \le \mu < 2$ the situation is the same because in this case $p = \bar{q}$, so that

$$\frac{b}{2}(\sinh v + \sinh q)\ln\frac{\cosh\frac{v-q}{2}}{\sinh\frac{v+q}{2}} + \frac{b}{2}(\sinh v + \sinh p)\ln\frac{\cosh\frac{v-p}{2}}{\sinh\frac{v+p}{2}} \in \mathbb{R} \tag{4.67}$$

if $v > -\sinh^{-1}\frac{a}{b}$. Therefore, $B = \frac{\pi}{2}(z - \frac{\mu}{2})$. For $v < -\sinh^{-1}\frac{a}{b}$, we have to replace $i$ by $-i$ in the logarithmic terms of (4.64) so that $B = -\frac{\pi}{2}(z + \frac{\mu}{2})$.

To study $B$ on the horizontal line $l = \{v : \Im v = \frac{\pi}{2}\}$, we first compute the derivative

$$\begin{aligned}\frac{\partial h}{\partial v}(v,p,q) &= \Bigg[(\mu - b(\cosh q + \cosh p))\cosh v \\ &\quad + \frac{b}{2}\ln\frac{\cosh\frac{q-p}{2} + \cosh(v - \frac{q+p}{2})}{\cosh\frac{q-p}{2} - \cosh(v + \frac{q+p}{2})}\Bigg]\cosh v \\ &= \tau(v,p,q)\cosh v\,.\end{aligned} \tag{4.68}$$

It is easy to see that $\tau(\frac{i\pi}{2}, p, q) = 0$ for all $p$ and $q$, so that $h$ has at least a third-order zero at $\frac{i\pi}{2}$. In the particular case $\mu = 2$, the previous equation becomes

$$\begin{aligned}\frac{\partial h}{\partial v}(v,u) &= \left[2(1 - b\cosh u)\cosh v + b\ln i\frac{\cosh\frac{v-u}{2}}{\sinh\frac{v+u}{2}}\right]\cosh v \\ &= \tau(v,u)\cosh v\,.\end{aligned} \tag{4.69}$$

Substituting $v = \frac{i\pi}{2} + \xi$ in the case $\mu < 2$, we get

$$\begin{aligned}\frac{\partial h}{\partial v} &= -\sinh\xi\Bigg[(\mu - b(\cosh q + \cosh p))\sinh\xi \\ &\quad - i\frac{b}{2}\ln\frac{\cos\Im q + i\sinh(\xi - \Re q)}{\cos\Im q - i\sinh(\xi + \Re q)}\Bigg].\end{aligned} \tag{4.70}$$

This expression becomes real for all $\xi$ if and only if $\Re q = 0$, i.e., if $a = 0$. According to (4.9), this implies $u = 0$ and, consequently, $x = 0$. Thus, for $x = 0$, $\frac{\partial h}{\partial v} = A_v + iB_v \in \mathbb{R}$ on $l$. That means $B$ is constant on $l$, and because $v = \frac{i\pi}{2}$ is a zero of $h$, we conclude that $l$ is a zero level curve for $B$. The same is true for the case $\mu \geq 2$, because in this case the logarithmic term of (4.66) becomes

$$\ln\frac{\cosh\frac{q-p}{2} + i\sinh(\xi - u)}{\cosh\frac{q-p}{2} - i\sinh(\xi + u)}\,, \tag{4.71}$$

where, according to (4.7), $u = \frac{q+p}{2}$.

To find other zero curves of $B$, we have to find zeros of $\frac{\partial h}{\partial v}$ on $l$ other than $\frac{i\pi}{2}$. First notice that $\frac{\partial h}{\partial v}(\frac{i\pi}{2} + \xi, q, p)$ is an even function of $\xi$ and that it goes to $+\infty$ as $\xi \to +\infty$ when $t > 0$. The latter fact follows from

$$\mu - b(\cosh q + \cosh p) = -4tb^2 < 0\,.$$

Direct computations yield

$$\begin{aligned}\frac{\partial \tau}{\partial v} &= [\mu - b(\cosh q + \cosh p)]\sinh v \\ &\quad - \frac{b}{2}\left[\frac{\cosh q}{\sinh v + \sinh q} + \frac{\cosh p}{\sinh v + \sinh p}\right].\end{aligned} \tag{4.72}$$

In the case $x = 0$ we know $a = 0$ and hence $p = -q$. Thus, the latter equation becomes

$$\frac{\partial \tau}{\partial v} = \left[\mu - 2b\cosh q + \frac{b\cosh q}{\cosh^2 q - \cosh^2 v}\right]\sinh v\,. \tag{4.73}$$

The equation $\frac{\partial \tau}{\partial v}(\frac{i\pi}{2} + \xi, q) = 0$ then becomes

$$\mu = b\cosh q\left[2 - \frac{1}{\cosh^2 q + \sinh^2 \xi}\right]. \tag{4.74}$$

Thus,

$$\sinh^2 \xi = \frac{\cosh q}{2b\cosh q - \mu}[b - 2b\cosh^2 q + \mu\cosh q]\,. \tag{4.75}$$

In order to show the existence of real solutions, we have to show that the right-hand side is positive for all $\mu \geq 0$. Substituting $b\cosh q = \sqrt{T^2 + b^2}$ and taking into account (4.12), the right-hand side of (4.75) becomes

$$\frac{\sqrt{T^2+b^2}}{b^2}\left[1 - 4t\sqrt{T^2+b^2}\,\right]. \tag{4.76}$$

Since $b(t)$ is an increasing function on $[0, t_0]$, where $t_0 = \frac{1}{2(\mu+2)}$ and $b(t_0) = \sqrt{\mu+2}$, it is clear that (4.76) is positive when $t \in [0, t_0)$ and is zero when $t = t_0$. (In the special case when $\mu = 0$, (4.76) also becomes 0 when $t = 0$.) Let $\xi_1 \geq 0$ denote the solution to (4.75). This is a simple zero of $\frac{\partial h}{\partial v}(\frac{i\pi}{2} + \xi, q)$.

The obtained results show that $A_\xi = \Re \frac{\partial h}{\partial v}(\frac{i\pi}{2} + \xi, q)$ is zero at $\xi = 0$; then it decreases until $\xi = \xi_1$ and afterwards increases towards $+\infty$. Thus, for any $\mu > 0$ and for any $t \in (0, t_0)$, there is a unique value $\xi_0 > 0$ such that

$$\frac{\partial h}{\partial v}\left(\frac{i\pi}{2} + \xi_0, q\right) = 0\,. \tag{4.77}$$

In the case $t = t_0$, the value $\xi_0 = 0$, thus forming a fifth-order zero of $h(v, q)$ at $v = \frac{i\pi}{2}$. Because $A_\xi = B_\eta$, where $v = \xi + i\eta$, we conclude that $B(v)$ changes sign: from negative to positive if $\xi > \xi_0$ as $v = \xi + i\eta$ crosses $l$ from below, and from positive to negative if $\xi \in (0, \xi_0)$. Since $B$ is positive along the real axis for $v > v_{+1}$, there is a zero level curve of $B$ in $S$ that goes to infinity. Simple asymptotic analysis of $B$ shows that there is only one such a curve. Thus, we have four zero level curves $c_j$, $j = 1, 2, 3, 4$, of $B$ inside the right half of the strip $S$, emanating from $\frac{i\pi}{2} + \xi_0$, $\frac{i\pi}{2}$, $v_{+1}$, and $\infty$, respectively; see Figure 4.6. The curve $c_2$ constitutes an angle $\frac{\pi}{3}$ with $l$ because $\frac{i\pi}{2}$ is a third-order zero of $h$.

Finally, let us study the behavior of $B$ along the imaginary segment $[0, \frac{i\pi}{2}]$. In the case $x = 0$ we have $q = -p$, so (4.68) becomes

$$\begin{aligned}\frac{\partial h}{\partial v}(v, p, q) &= \left[(\mu - 2b\cosh q)\cosh v + \frac{b}{2}\ln\frac{\cosh q + \cosh v}{\cosh q - \cosh v}\right]\cosh v \\ &= \tau(v, q)\cosh v\,.\end{aligned} \tag{4.78}$$

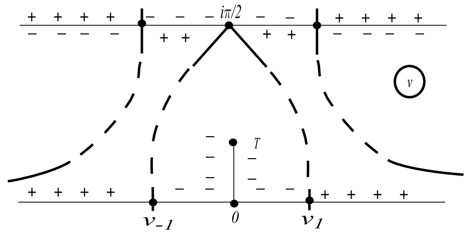


FIGURE 4.6. Zero level curves, $x = 0$.

We want to show that $B \leq 0$ on $[0, \frac{i\pi}{2}]$. Since $B(0, \frac{\pi}{2}) = 0$, it is enough to show that $B_\eta = A_\xi$, where $v = \xi + i\eta$, is nonnegative on $[0, \frac{i\pi}{2}]$ if $\mu \geq 2$ and on $[q, \frac{i\pi}{2}]$ if $0 \leq \mu < 2$. Thus, we want to show that $M$, where $\tau = M + iN$, is nonnegative there. Since $\tau(\frac{i\pi}{2}, q) = 0$, it is enough to show that $N_\xi = -M_\eta$ is nonnegative. Similarly to (4.73), we find

$$N_\xi(0, \eta) = \left[\mu - 2b\cosh q + \frac{b\cosh q}{\cosh^2 q - \cos^2\eta}\right]\sin\eta\,. \tag{4.79}$$

As before, the equation $N_\xi(0, \eta) = 0$ yields

$$-\cos^2\eta = \frac{\cosh q}{2b\cosh q - \mu}[b - 2b\cosh^2 q + \mu\cosh q]\,. \tag{4.80}$$

But, as shown above, the right-hand side of (4.80) is nonnegative. That implies that $N_\xi(0, \eta) \geq 0$ on $[0, \frac{i\pi}{2}]$ if $\mu \geq 2$ and on $[q, \frac{i\pi}{2}]$ if $0 \leq \mu < 2$. It is clear that in the latter case $N_\xi(0, \eta) < 0$ on $[0, q]$. That implies $B_\eta > 0$ on $[0, \frac{i\pi}{2}]$ except, possibly, some segment $[0, s] \subset [0, q)$. But because $B(0, 0) < 0$, we again obtain that $B$ is negative on the whole $[0, \frac{i\pi}{2}]$. Thus, zero level curves of $B$ cannot cross the semi-interval $[0, \frac{i\pi}{2})$.

Since $B(\xi, \eta)$ is a harmonic function, every closed and bounded level curve of $B$ should contain at least one singular point. Thus, the zero level curve $c_2$ emanating into $S$ from $\frac{i\pi}{2}$ cannot end at $\frac{i\pi}{2} + \xi_0$. It also cannot end at $\infty$, since then it would be intersected by $c_1$. As the only remaining option, $c_2$ connects $\frac{i\pi}{2}$ and $v_{+1}$, whereas $c_1 \subset S$ goes from $\frac{i\pi}{2} + \xi_0$ to $\infty$ without intersecting $c_2$. Therefore, $c_2$ is the image of our contour $\gamma^+$ under the map $z \mapsto v$. It is also clear that $A = \Re h$ is decreasing along $\gamma^+$. The existence of the part of the contour $\gamma$ connecting $\alpha$ and $-\frac{\mu}{2}$ follows from the fact that there is a zero level curve of $B$ connecting $\frac{i\pi}{2}$ and $v_{-1}$. Thus, the existence of the contour $\gamma$ satisfying all conditions of (2.19) is established for $x = 0$. The proof is completed. $\square$

## 4.6 Existence of the Contour $\gamma$ When $x > 0$

THEOREM 4.10 *Let $t_0(x)$ be the maximal time such that all the points $(x, t)$, where $x \geq 0$ is fixed and $t \in [0, t_0(x))$, belong to the genus 0 region. Then $t_0(x) > 0$ for all $x \geq 0$.*

PROOF: In the case $x = 0$, the existence of $t_0(0) = \frac{1}{2(\mu+2)}$ was proven above. In the case $x > 0$ the horizontal line $l$ is not a zero level curve of $B$ anymore. However, $B$ is monotonically decreasing along $l$ according to (4.70) and (4.71), where $u > 0$. The fact that $B(0, \frac{\pi}{2}) = 0$ implies $B < 0$ for $\xi > 0$ and $B > 0$ for $\xi < 0$.

Consider now the asymptotic behavior of $B(\xi, \eta)$ in $S$ when $\xi \to \pm\infty$. Considering the first logarithmic term of (4.64), we obtain

$$\ln i \frac{\cosh \frac{v-q}{2}}{\sinh \frac{v+q}{2}} \sim \pm \frac{i\pi}{2} \mp q + 2e^{\mp v} \cosh q + O(e^{\mp 2v}) \tag{4.81}$$

as $v \to \pm\infty$. Substituting this and the corresponding expression for the other logarithmic term and taking into account (4.7), we obtain

$$\begin{aligned} h(v, p, q) \sim &\mp \frac{1}{2} t b^2 e^{\pm 2v} \\ &\pm \frac{b}{4} e^{\pm v} [\pm i\pi \mp (q+p) + 2e^{\mp v}(\cosh q + \cosh p)] \\ &\pm \frac{i\pi}{2} \frac{b}{2} (\sinh q + \sinh p) - \frac{\mu}{2}\Big(\frac{i\pi}{2} - v\Big) + O(e^{\mp v}) \end{aligned} \tag{4.82}$$

as $v \to \pm\infty$. Substitution of $v = \xi + i\eta$ together with (4.7) yields

$$B(\xi, \eta) \sim -\frac{1}{2} t b^2 e^{2\xi} \sin 2\eta + \frac{b}{4} e^{\xi} (\pi \cos\eta - 2u \sin\eta) + O(\xi) \tag{4.83}$$

if $\xi \to +\infty$ and

$$B(\xi, \eta) \sim -\frac{1}{2} t b^2 e^{-2\xi} \sin 2\eta + \frac{b}{4} e^{-\xi} (\pi \cos\eta + 2u \sin\eta) + O(\xi) \tag{4.84}$$

if $\xi \to -\infty$ in $S$. Notice that on the boundary lines $\eta = 0$ and $\eta = \frac{\pi}{2}$ of $S$, the function $B$ attains correspondingly positive and negative values as $\xi \to +\infty$, in accordance with our previous analysis. Moreover, according to (4.83), there exists a unique zero level curve of $B$ in $S$ that is asymptotic to $2tbe^{\xi} \sin 2\eta = \pi \cos\eta - 2u \sin\eta$ as $\xi \to +\infty$. In particular, this equation asymptotically approaches the graph $\eta = \pi e^{-\xi}/(4tb)$ if $t > 0$ and $\eta = \tan^{-1} \frac{\pi}{2x}$ if $t = 0$.

In the left half of $S$, however, according to (4.84), the function $B > 0$ for all $\eta \in [0, \frac{\pi}{2}]$ if $t = 0$ and $-\xi$ is sufficiently large. So, for $t = 0$ there are no zero level curves of $B$ going to infinity in the left half of $S$. Since the three zero level curves of $B$ emanating from $\frac{i\pi}{2}$ into $S$ cannot intersect each other, they have to connect $\frac{i\pi}{2}$ with the points $v_{-1}$, $v_{+1}$, and $+\infty$. Thus, for $t = 0$ the existence of contour $\gamma^+$ is proven. Consider now small but positive $t$. Then, according to (4.84), $B < 0$ as $\xi \to -\infty$ along any horizontal line $\eta = \text{const}$ except $\eta = 0, \frac{\pi}{2}$. Along these boundaries of $S$ the function $B$ is positive. Thus, there are two zero level curves of $B$ that are asymptotic to $\eta = 0, \frac{\pi}{2}$ as $\xi \to -\infty$. If $t$ is small, these curves meet each other and form a single curve $c$, given asymptotically by $2tbe^{-\xi} \sin 2\eta = \pi \cos\eta + 2u \sin\eta$. Thus, the other zero level curves of $B$ are

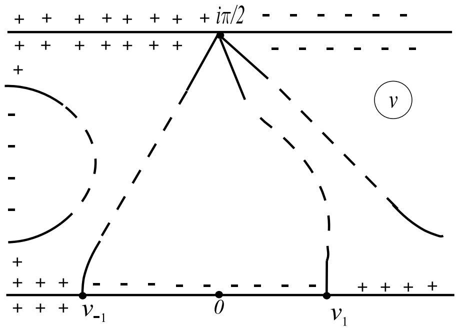


FIGURE 4.7. Zero level curves, $x > 0, t < t_0(x)$.

topologically the same as for the case $t = 0$, and so the contour $\gamma^+$ still exists; see Figure 4.7.

Let us now show that the presence of the vertical cut $[0, T]$ in the case $0 < \mu < 2$ (function $h(z)$ has singularities of the type $z \ln z$ at $z = \pm T$) will not change the topology of zero level curves of $B = \Im h$ established above. We first prove that $B(0, y)$, where $z = \zeta + iy$, is a convex function on the cut $[0, T]$.

LEMMA 4.11 *For every $x \geq 0$, $t \geq 0$, the function $B(0, y)$, where $h = A + iB$, is convex on $y \in [0, |T|]$. If $t = 0$, then, additionally, $B(0, y)$ is concave on $y \in [|T|, +\infty)$.*

PROOF: We start with the observation that $h_z = \tau$ and $h_{zz} = \frac{\tau_v}{b \cosh v}$, where $z - a = b \sinh v$. Since $h_{zz} = A_{\zeta\zeta} + iB_{\zeta\zeta} = A_{\zeta\zeta} - iB_{yy}$, we need to show that the statement $B_{yy} > 0$ is equivalent to $\Im h_{zz} = \Im \cosh \bar{v} \tau_v < 0$. Notice that $\Im h(z)$ is continuous across the cut $[0, T]$ (but $\Re h(z)$ has a jump there). Let $z = iy$ be a point on $[0, T)$. Then the corresponding $\sinh v + \sinh p = \frac{iy - T}{b}$, $\sinh v + \sinh q = \frac{iy + T}{b}$. Then (4.72) together with (4.7) yields

$$\frac{\partial \tau}{\partial v} = \left[ -4tb^2 - \frac{1}{2} b^2 \frac{\mu + 4tb^2}{|T|^2 - y^2} \right] \sinh v - \frac{b^3 \sinh 2u}{2(|T|^2 - y^2)} . \tag{4.85}$$

Thus, $\tau_v$ is a linear function in $\sinh v$, where both coefficients are negative.

The proof now follows from the fact that both

$$\Im \cosh \bar{v} = -\sinh \xi \sin \eta \quad \text{and} \quad \Im \cosh \bar{v} \sinh v = \frac{1}{2} \sin 2\eta , \tag{4.86}$$

where $v = \xi + i\eta$, are positive if $\sinh(\xi + i\eta) = \frac{iy - a}{b}$. □

Since $B(0, 0) < 0$, Lemma 4.11 implies that either $B < 0$ everywhere on $[0, T]$, or the cut is transversally intersected with only one zero level curve of $B$. Let us now show that for $t = 0$ only a complementary arc $\gamma_c$ connecting $\alpha$ to $-\frac{\mu}{2}$

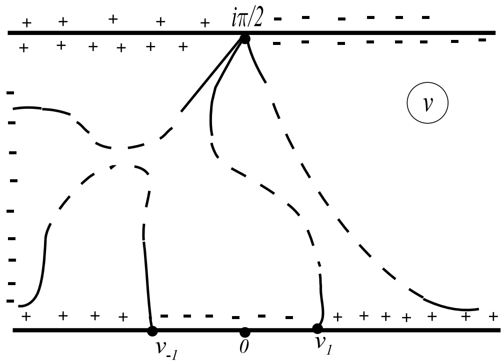


FIGURE 4.8. Zero level curves, $x > 0$, $t > t_0(x)$.

can cross the cut $[0, T]$. Indeed, at $x = 0$ no zero level curve of $\Im h$ intersects the cut; see Section 4.5. Let us vary $x$. According to (4.64),

$$\Im h(T) = |T| \ln \frac{\cosh x}{|T|} + \frac{\mu}{2}\left(\sin^{-1}|T| - \frac{\pi}{2}\right). \tag{4.87}$$

We see that $\Im h(T) > 0$ for sufficiently large $x$, so there will be an intersection with the cut. If the main arc crosses the cut, it should also cross the imaginary axis above $T$. Thus, the interval $(T, i\infty)$ will be crossed at least twice by the main and the complementary arcs. But that contradicts the concavity of $\Im h$ on $(T, i\infty)$. Now fix some $x > 0$ and let $t$ grow. If $\Im h(T) < 0$ at $t = 0$, then $\Im h$ will be negative on the cut $[0, T]$ for all $t$, since $\Im h_t < 0$ there; see Lemma 5.3. If $\Im h(T) \geq 0$ at $t = 0$, then the complementary arc has already crossed $[0, T]$, and no other zero level curve of $\Im h$ can cross the cut for $t > 0$.

In the case when the complementary arc $\gamma_c$ intersects $[0, T]$ (then, obviously, $\Im h(T) \geq 0$), we can always deform $\gamma_c$ within the positive domain of $\Im h$ (i.e., domain $\Im h(z) > 0$) so that $\gamma_c$ goes around $z = T$ and does not cross the cut. Thus, conditions (2.19) will be preserved, as well as the topology of zero level curves of $\Im h$ discussed above.

If $x > 0$ is kept fixed and $t$ increases, the only way to change the topological picture of the level curves of $B$ is for the curve $c$ to collide with the level curve connecting $\frac{i\pi}{2}$ and $v_{-1}$ at some $t_0(x) > 0$.

It is clear that the condition $\Im(g_+ + g_- - f) > 0$ when $z \in \gamma_c^+$ of (2.19) with $N = 0$ fails at the point $(x, t_0(x))$. The topology of zero level curves of $\Im h$ for $t > t_0(x)$ is given in Figure 4.8. Thus, the theorem is established. □

As we have seen in the proof of Theorem 4.10, the point $(x, t_0(x))$:

(1) satisfies the system of equations

$$\tau(v, p, q) = 0 \quad \text{and} \quad \Im h(v, p, q) = 0, \tag{4.88}$$

and

(2) all points $(x, t)$, where $t \in [0, t_0(x))$, belong to the genus zero region.

The set of all points in the $(x, t)$–plane satisfying (1) and (2) is called a *breaking curve*, and each such point is called a *breaking point*. In particular, it is clear that $x = 0$ and $t = \frac{1}{2(\mu+2)}$ is a breaking point. The existence and properties of the breaking curve are discussed in the next section.

# 5 Breaking Curve

In this section we prove that for any $x > 0$ there exists a finite value $t_0(x)$ such that the conditions (2.19) with $N = 0$ are satisfied for all $t \in [0, t_0(x))$ but fail at $t = t_0(x)$. Thus, the function $t_0(x)$ is defined for every $x \geq 0$. We prove that $t_0(x)$ is a smooth, monotonically increasing, single-valued function that has linear behavior as $x \to 0$ and $x \to \infty$.

## 5.1 Existence of the Breaking Curve

As we have seen in Section 4.6, system (4.88) determines a breaking point for a given $x > 0$. Although we cannot solve the system (4.88) in explicit form, we can show that such a point exists for any $x > 0$. Namely, we show that there exists a line $l$ in the upper $z$ half-plane separating $\alpha$ and $-\frac{\mu}{2}$ such that $\Im h < 0$ along $l$ for sufficiently large $t$. Thus, conditions (2.19) with $N = 0$ fail for sufficiently large $t$ since there is no zero level curve of $h$ connecting $\alpha$ and $-\frac{\mu}{2}$.

LEMMA 5.1 *For every $x \geq 0$ there exists a finite $t_0(x)$.*

PROOF: Using (4.68) and the fact that $\frac{dh}{dz} = \frac{dh}{dv} \cdot \frac{1}{b \cosh v} = b^{-1}\tau(v, p, q)$, we calculate

$$\frac{dh}{dz} = \frac{1}{2}\left[\ln \frac{a + R(z)\sinh u \cosh u - (z-a)\sinh^2 u}{a - R(z)\sinh u \cosh u - (z-a)\sinh^2 u} - 8tR(z)\right]. \tag{5.1}$$

Let us fix some $\xi \in (-\frac{\mu}{2}, 0)$ and consider the vertical ray $l = \{z : z = \xi + i\eta,\ \eta \geq 0\}$. We want to show that $B = \Im h < 0$ on $l$ for a fixed $x$ and sufficiently large $t$. Since, according to (4.66)–(4.67), we have $B(\xi, 0) < 0$, it is sufficient to prove $B_\eta = A_\xi = \Re \frac{dh}{dz} < 0$ on $l$; see Figure 5.1.

Taking into account Corollary 4.2, we obtain

$$\begin{gathered} R(z) \sim (a - z) + \frac{b^2}{2(a-z)}\,, \\ \sqrt{(a \pm T)^2 + b^2} \sim (a \pm T) + \frac{b^2}{2(a \pm T)}\,, \quad t \to \infty\,, \end{gathered} \tag{5.2}$$

uniformly in $\Re z < 0$. Therefore, using (4.10), we obtain

$$R(z)\sinh u \cosh u = (a - z)\sinh u \cosh u + \frac{1}{2(a-z)} + o(1) \tag{5.3}$$

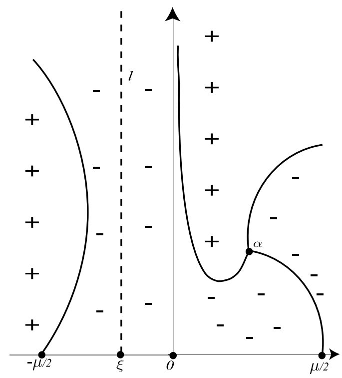


FIGURE 5.1. Signs of $\Im h$ for large $t$.

as $t \to \infty$. The real part of the logarithm in (5.1) then becomes

$$\ln \frac{|(a-z)e^{2u} + (a+z) + \frac{1}{(a-z)} + o(1)|}{|(a+z) + \frac{1}{(a-z)} + o(1)|} = 2u + \ln \frac{|(a-z) + o(1)|}{|(a+z) + \frac{1}{(a-z)} + o(1)|} . \tag{5.4}$$

The latter logarithm is uniformly bounded on $l$ for all large $t$. Thus, using (4.11), we see that

$$\Re \frac{dh}{dz} = (4ta + x) - 4t(a - \xi) + Q = 4t\xi + x + Q < 0 \tag{5.5}$$

since $t$ is sufficiently large and $Q$ is bounded. Thus, the contour $\gamma_c^+$ in conditions (2.19) cannot exist if $t$ is sufficiently large, and the proof is complete. □

## 5.2 Properties of the Breaking Curve

Here we show that the function $t_0(x)$ is smooth and monotonically increasing for all positive $x$. Let $v_0(x)$, together with $t_0(x)$, denote a solution of the system (4.88) for a given $x \geq 0$, and let $z_0(x)$ denote the point corresponding to $v_0(x)$ on the complex $z$-plane. The following three lemmas are needed to prove the above-mentioned properties of the breaking curve $t_0(x)$.

LEMMA 5.2 *If $\mu \geq 2$ and $(x, t_0(x))$ is a breaking point, then the corresponding $\Re z_0(x) \leq 0$. Moreover, $\Re z_0(x) = 0$ implies $x = 0$. If $\alpha \to \frac{\mu}{2}$, then $z_0$ has to approach the interval $(-\infty, -T)$ if $\mu \geq 2$ or the union $(-\infty, 0] \cup [0, T]$ if $\mu < 2$.*

The proof of the lemma is based on MI conditions (3.5)–(3.9) for $N = 1$ and is given in Section 6.4. It does not use any results concerning the breaking curve obtained in the present section.

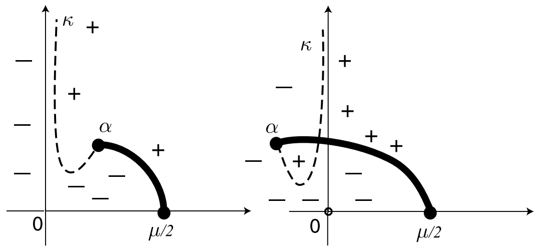


FIGURE 5.2. Signs of $\Im h_t$.

LEMMA 5.3 *The whole zero level curve* $\kappa$ *of* $\Im h_t$ *from Lemma* 4.5 *lies in the same half-plane* (*left or right*) *where the point* $\alpha \in \kappa$ *lies*; *see Figure* 5.2.

PROOF: We first express $h_t$, given by (4.46), in the hyperbolic variable $v = \xi + i\eta$ as

$$h_t(v, x, t) = -2b^2[\sinh v + \sinh p + \sinh q] \cosh v\,. \tag{5.6}$$

Then equation $\Im h_t = 0$ yields the solution

$$\cos\eta = -\frac{a}{b}\frac{\sinh\xi}{\sinh^2\xi + \frac{1}{2}}\,. \tag{5.7}$$

Therefore, according to (4.6) and (4.63),

$$\Re z = b\Re\sinh v + a = b\sinh\xi\cos\eta + a = \frac{a}{\cosh 2\xi}\,, \tag{5.8}$$

so that the sign of $\Re z$ coincides with that of $a$. The proof is complete. □

LEMMA 5.4 *For any* $x > 0$, *we have* $\Im h_x(z_0(x)) > 0$ *and* $\Im h_t(z_0(x)) < 0$.

PROOF: In the case $\mu \geq 2$ the point $z_0(x)$ is in the left half-plane, according to Lemma 5.2. Moreover, since $z_0(x) \in \gamma_c^+$, there exists a continuous curve in the upper half-plane that connects $z_0(x)$ with $-\infty$ and does not cross $\gamma^+$. Thus, for $\mu \geq 2$, the statement of the lemma follows from Lemma 4.5.

In the case $\mu < 2$ we first consider $\Im h_t$. According to (5.7), the zero level curve $\kappa$ of $\Im h_t(v)$ lies in the left $v$ half-plane. Therefore, the assumption $\Re v_0 = \xi_0 \geq 0$, combined with Lemma 4.5 and the fact that $z_0(x) \in \gamma_c^+$, immediately yields $\Im h_t(z_0(x)) < 0$. So, we consider only $\xi_0 < 0$. We also exclude the case $\Im v_0 = \eta_0 = \frac{\pi}{2}$, since, according to the results of Sections 4.5 and 4.6 (namely, that $\Im h(v_0) = 0$ and $\eta_0 = \frac{\pi}{2}$ implies $v_0 = \frac{i\pi}{2}$), that would imply $x = 0$. Solving the equation $h_v = 0$ for the logarithmic term

$$\frac{1}{2}b\left[\ln i\frac{\cosh\frac{v-q}{2}}{\sinh\frac{v+q}{2}} + \ln i\frac{\cosh\frac{v-p}{2}}{\sinh\frac{v+p}{2}}\right] = -[\mu - b(\cosh q + \cosh p)]\cosh v \tag{5.9}$$

and substituting it into the other equation $\Im h = 0$ of (4.88), we obtain after some algebra

$$(5.10)\quad \begin{aligned}\Im h = &-\frac{1}{2}[\mu - b(\cosh q + \cosh p)]\Im[(\sinh v + \sinh p + \sinh q)\cosh v] \\ &+ \frac{1}{2}\Im\left[T \ln \frac{\cosh\frac{v-q}{2}\sinh\frac{v+p}{2}}{\cosh\frac{v-p}{2}\sinh\frac{v+q}{2}}\right] + \frac{\mu}{2}\Im\Big(v - \frac{i\pi}{2}\Big) = 0\,.\end{aligned}$$

The latter term of $\Im h(v_0)$ is negative since $\eta_0 \in [0, \frac{\pi}{2})$. Let us show that the logarithmic term in (5.9) cannot be positive if $\xi = \Re v < 0$. Indeed, $\mu < 2$ implies $T = i|T|$, so we need to show that

$$(5.11)\quad \left|\frac{\sinh u + \sinh(v - \frac{q-p}{2})}{\sinh u + \sinh(v + \frac{q-p}{2})}\right| \le 1\,.$$

Note that, according to the last equation of (4.11), $b \sinh u > a$ if $m < 2$. Therefore, using (4.8) and (4.9), we obtain $s \in [0, \frac{\pi}{2}]$ where

$$s = -\frac{i}{2}(q - p) = -i \cosh^{-1} \frac{a}{b \sinh u}\,.$$

Note that for any $\eta, s \in [0, \frac{\pi}{2}]$, we have $\sin^2(\eta + s) \ge \sin^2(\eta - s)$ and $\cos(\eta + s) \le \cos(\eta - s)$. To prove inequality (5.11), it remains to reduce it to

$$(5.12)\quad \begin{aligned}&\sin^2(\eta - s) + 2 \sinh u \sinh \xi \cos(\eta - s) \le \\ &\qquad \sin^2(\eta + s) + 2 \sinh u \sinh \xi \cos(\eta + s)\,,\end{aligned}$$

which is true since $\xi < 0$. Now we see that (5.10) implies $\Im[(\sinh v + \sinh p + \sinh q)\cosh v] > 0$, since $-\mu + b(\cosh q + \cosh p) = 4tb^2 > 0$. Thus, the required inequality $\Im h_t(v_0(x)) < 0$ follows from (5.6). Notice that in the case $\mu = 2$, the proof is still valid.

It remains to consider $\Im h_x(v_0(x))$ in the case $\mu < 2$. We first show that equation $h_v = 0$ of (4.88) is incompatible with $\Im h_x(v) = b \Im \cosh v = 0$. Indeed, the latter equation implies $v = i\eta$, where $\eta \in [0, \pi)$. So, (5.9) becomes

$$(5.13)\quad \frac{\cosh \frac{q-p}{2} + \cosh(u - i\eta)}{\cosh \frac{q-p}{2} - \cosh(u + i\eta)} = e^{8tb \cos \eta}\,.$$

The requirement that the left-hand side of (5.13) is real yields

$$\omega^2 - \cosh^2(u - i\eta) \in \mathbb{R}\,,$$

where $\omega = \cosh \frac{q-p}{2}$ is real. Thus, $\cosh(u - i\eta)$ should be either real or purely imaginary, so that $\eta = 0$ or $\eta = \frac{\pi}{2}$ or $u = 0$. The first condition is incompatible with $\Im h(v) = 0$, whereas any of the other conditions combined with $v = v_0(x)$ implies $x = 0$. Thus, $\Im h_x(v_0(x)) \ne 0$. To complete the proof of the lemma, one has to note that, according to Corollary 4.2, Lemma 4.5, and Lemma 5.2, $\Im h_x(v_0(x)) > 0$ for sufficiently large $x$. □

THEOREM 5.5 *The function $t_0(x)$ is smooth and monotonically increasing for all $x > 0$.*

PROOF: Let $t_0 = t_0(x_0)$, $x_0 > 0$, be a breaking point. That means that the system of equations (4.88) is satisfied with the given $(x_0, t_0)$ and some uniquely determined $v_0 = v_0(x_0, t_0)$. (It is easy to see that the assumption of several corresponding $v_0$ leads to a contradiction, since it would imply existence of a closed zero level curve of the harmonic function $\Im h(v)$.)

Let us fix $x = x_0$ and write (4.88) in vector form as $F(v, x, t) = 0$, where the first two components of $F$ are $\Re\tau$ and $\Im\tau$, and the last component is $\Im h$. Using the Cauchy-Riemann equation, we calculate the Jacobian

$$\left|\frac{\partial F}{\partial(v,t)}\right| = \left|\frac{\partial \tau}{\partial v}\right|^2 \cdot \Im h_t\,, \tag{5.14}$$

since $h_v = 0$ implies $\frac{\partial \Im h}{\partial \Re v} = \frac{\partial \Im h}{\partial \Im v} = 0$. Based on the topology of zero level curves of $\Im h$, considered in Section 4.6, we note that $h_{vv}(v_0) \neq 0$ for any $v_0$, and hence $\tau_v(v_0) \neq 0$ at this point. The fact that $\Im h_t(v_0) < 0$ follows from Lemma 5.4. Thus, according to the implicit function theorem, the system $F(v, x, t) = 0$ uniquely defines a smooth real function $t = t_0(x)$ and a smooth, complex-valued function $v = v_0(x)$ in a neighborhood of $x = x_0$.

To prove that $t_0(x)$ is monotonically increasing, we calculate the full derivative

$$\frac{d}{dx}h = h_v\frac{dv}{dx} + h_t\frac{dt}{dx} + h_x \tag{5.15}$$

along the solution $(v_0(x), t_0(x))$ of $F(v, x, t) = 0$ and consider the imaginary part of (5.15). Then, according to Lemma 5.4, $\frac{dt}{dx} = -\frac{\Im h_x(v_0)}{\Im h_t(v_0)} > 0$. The proof is completed. □

## 5.3 Asymptotics of the Breaking Curve

Here we study the asymptotics of $t_0(x)$ for large and small positive $x$. The asymptotics obtained below near $x = 0$ shows continuity of $t_0(x)$ at $t_0(0) = \frac{1}{2(\mu+2)}$.

THEOREM 5.6 *For any $\mu > 0$,*

$$t_0(x) = \begin{cases} \dfrac{x}{2\mu}(1 + o(1)) & \text{as } x \to \infty \\ \dfrac{1}{2(\mu+2)} + \dfrac{\cot\frac{\pi}{5}}{2\sqrt{\mu+2}}x + O(x^{3/2}) & \text{as } x \to 0\,. \end{cases} \tag{5.16}$$

PROOF: We first prove the large $x$ asymptotics. According to Lemma 5.2 and Corollary 4.2, the asymptotics (5.2), where $z$ satisfies $h_z(z, x, t) = 0$, is valid for all large $x$ uniformly in $t \geq 0$. Substituting (5.2) into system (4.88), namely, into equations (4.41) and (5.1), and omitting some lower-order terms, we obtain

$$-\frac{\pi}{2}\Big(\frac{\mu}{2}-\Re z\Big)-\Im z\ln b-\Im\Big[\Big(\frac{\mu}{2}-z\Big)\ln(a-z)\Big] + \Im\Big[\frac{z}{2}\ln\frac{a^2-T^2}{z^2-T^2}\Big]+\Im\Big[\frac{T}{2}\ln\frac{(a+T)(z-T)}{(a-T)(z+T)}\Big]=0 \tag{5.17a}$$

and

$$2u+\ln\frac{a-z}{a+z+\frac{1}{2(a-z)}}=8t(a-z)+\frac{4tb^2}{a-z}\,. \tag{5.17b}$$

According to (4.9), the last term of the second equation is vanishing as $u\to\infty$. Then the real and imaginary part of the second equation can be written as

$$x=-4t\Re z \quad\text{and}\quad -8t\Im z=\arg(a-z)-\arg\Big(a+z+\frac{1}{2(a-z)}\Big). \tag{5.18}$$

Let us show that, according to Lemma 5.2, $\Im z\to 0$ as $x\to\infty$. In the case $\mu\geq 2$ the statement is clear. Suppose now that $\mu<2$ and $z$ approaches $[0,T]$ as $x\to\infty$. Then, according to the first equation, $t\to\infty$. But the right-hand side of the second equation is bounded, so $\Im z\to 0$. Therefore, $z$ approaches the negative real semi-axis as $x\to\infty$ for any $\mu>0$.

According to (4.9), $t\sim b^{-2}$ as $u\to\infty$. Thus, boundedness of $t\Im z$ implies $\Im z\ln b\to 0$ as $x\to\infty$. Collecting the leading-order terms of equation (5.17a), we obtain

$$-\frac{\pi}{2}\Big(\frac{\mu}{2}-\Re z\Big)-\frac{1}{2}\Re z\arg(z^2-T^2)=0 \tag{5.19}$$

as $x\to\infty$. Thus, $\Re z\to-\frac{\mu}{2}$ as $x\to\infty$. Substituting this expression into the first equation of (5.18) yields (5.16). The proof of the large $x$ asymptotics is complete.

Since we know that $t_0(x)$ is a smooth, increasing function on $(0,\infty)$, we are looking for a solution $(z_0(x),t_0(x))$ to (4.88), where $t_0(x)$ is a smooth growing solution for small $x>0$. We want to find such a solution under the assumption that $z_0(x)$ is close to $z_0(0)=i\sqrt{\mu+2}$, where the breaking for $x=0$ occurs. Based on results of Section 4.1, this assumption implies that the values $a(x)$, $b(x)$, and $t(x)=t_0(x)$ are close to 0, $\sqrt{\mu+2}$, and $\frac{1}{2(\mu+2)}$, correspondingly, and that $u=4ta+x\to 0$ as $x\to 0$. Taking the two leading terms of the Taylor expansion of $\tau(v,p,q)$ at $v=\frac{i\pi}{2}$, we rewrite the system (4.88) for the breaking curve $t_0(x)$ for small $x>0$ as

$$\begin{aligned}\tau_v\Big(\frac{i\pi}{2},p,q\Big)+\tau_{vvv}\Big(\frac{i\pi}{2},p,q\Big)\frac{\zeta^2}{6}&=0\,,\\ \Im[h_{vvv}\Big(\frac{i\pi}{2},p,q\Big)\zeta^3]&=0\,,\end{aligned} \tag{5.20}$$

where $v=\frac{i\pi}{2}+\zeta$, $\Im\zeta<0$, and $\zeta\in\mathbb{C}$ is small. The simple facts that $\tau_{vv}(\frac{i\pi}{2},p,q)=0$ and $h_{vvv}(\frac{i\pi}{2},p,q)=2i\tau_v(\frac{i\pi}{2},p,q)$ are used here and below.

We first consider the particular case $\mu = 2$ and then make required adjustments for the general case. Since in this case $p = q = u$, we get

$$\frac{\partial \tau}{\partial v} = 2(1 - b\cosh u)\sinh v - \frac{b\cosh u}{\sinh v + \sinh u}\,. \tag{5.21}$$

Then, using (4.29), we obtain

$$\frac{\partial \tau}{\partial v}\left(\frac{i\pi}{2}, u\right) = \frac{2}{s(u,x)\cosh u}[-\sinh^2 u - i[(u-x)\cosh u - \sinh u]]\,, \tag{5.22}$$

where $s(u,x) = \sinh 2u - (u - x)$.

The complex number $\frac{\partial \tau}{\partial v}(\frac{i\pi}{2}, u)$ is in the second or third quadrant, so

$$\arg \frac{\partial \tau}{\partial v}\left(\frac{i\pi}{2}, u\right) = -\tan^{-1}\left(\frac{x}{u^2} + O(u)\right) + \pi \tag{5.23}$$

for small $u > 0$. Direct calculations for the third derivative yield

$$\begin{aligned} \tau_{vvv}(v,u) &= \\ &2(1 - b\cosh u)\sinh v - b\cosh u \frac{\cosh^2 v + 1 - \sinh v \sinh u}{(\sinh v + \sinh u)^3}\,, \end{aligned} \tag{5.24}$$

$$\begin{aligned} \tau_{vvv}\left(\frac{i\pi}{2}, u\right) = \frac{2}{s(u,x)\cosh^3 u}\big[&2\sinh^2 u - i[(u-x)\cosh^3 u \\ &+ (1 - \sinh^2 u)\sinh u]\big]\,, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} \arg \tau_{vvv}\left(\frac{i\pi}{2}, u\right) &= -\tan^{-1} \frac{(u-x)\cosh^3 u + (1 - \sinh^2 u)\sinh u}{2\sinh^2 u} \\ &= -\tan^{-1}\left(\frac{2u - x}{u^2} + O(u)\right), \end{aligned} \tag{5.26}$$

where $u \to 0$. Since $\Re\tau_{vvv}(\frac{i\pi}{2}, u) \to 0$ but $\Im\tau_{vvv}(\frac{i\pi}{2}, u)$ does not approach 0 as $u \to 0$, we see $\arg \tau_{vvv}(\frac{i\pi}{2}, u) \to -\frac{\pi}{2}$ as $u \to 0$. Moreover, the requirement

$$\zeta^2 = -6\frac{\tau_v(\frac{i\pi}{2}, u)}{\tau_{vvv}(\frac{i\pi}{2}, u)} \to 0 \quad \text{as } u \to 0$$

implies $\tau_v(\frac{i\pi}{2}, u) \to 0$, so that, according to (5.22), $x = o(u)$.

Equations (5.20) can now be written as

$$\begin{aligned} \arg\zeta &= \frac{1}{2}\arg\tau_v - \frac{1}{2}\arg\tau_{vvv} + \frac{\pi}{2} + \pi k\,, \\ 3\arg\zeta &= -\frac{\pi}{2} - \arg\tau_v + \pi n\,, \end{aligned} \tag{5.27}$$

where $k, n \in \mathbb{Z}$. The requirement $\Im\zeta < 0$, together with (5.23) and (5.26), yield $k = -2$ and $n = -1$. Thus, (5.27) can be rewritten as

$$\begin{aligned} \arg\zeta &= -\frac{1}{2}\tan^{-1}\left(\frac{x}{u^2} + O(u)\right) - \frac{1}{2}\arg\tau_{vvv} - \pi\,, \\ 3\arg\zeta &= -\frac{5}{2}\pi + \tan^{-1}\left(\frac{x}{u^2} + O(u)\right). \end{aligned} \tag{5.28}$$

The combination of these equations yields

$$5\tan^{-1}\left(\frac{x}{u^2} + O(u)\right) = -\pi + 3\tan^{-1}\left(\frac{2u - x}{u^2} + O(u)\right), \tag{5.29}$$

so that in the limit $u \to 0$ we obtain

$$\tan^{-1}\frac{x}{u^2} = \frac{\pi}{10}\,. \tag{5.30}$$

This equation shows that along the breaking curve $t_0(x)$ we have $x = \delta u^2$, where $\delta \to \tan\frac{\pi}{10}$ in the limit $x \to 0$.

Computing the leading behavior of $t$ as $u \to 0$, where $x = \delta u^2$, by (4.15), we obtain

$$t = \frac{1}{2(\mu+2)} + \frac{1 - \frac{1}{4}(\mu+2)\delta^2}{2(\mu+2)}u^2 + O(u^3)\,. \tag{5.31}$$

Substitution of $\mu = 2$ and $\delta = \tan\frac{\pi}{10}$ yields

$$t_0(x) = \frac{1}{8} + \frac{\cot\frac{\pi}{10} - \tan\frac{\pi}{10}}{8}x + O(x^{\frac{3}{2}})\,, \quad x \to 0\,. \tag{5.32}$$

Thus, we construct the smooth and growing solution $t_0(x)$ for small $x > 0$ in the case $\mu = 2$.

Let us now calculate $\tau_v(\frac{i\pi}{2}, p, q)$ and $\tau_{vvv}(\frac{i\pi}{2}, p, q)$ for general $\mu \geq 0$. Based on (4.72) and using (4.11), we obtain

$$\frac{\partial\tau}{\partial v}\left(\frac{i\pi}{2}\right) = [\mu - b(\cosh q + \cosh p)]i - \frac{b\sinh 2u - ib(\cosh q + \cosh p)}{2\cosh q\cosh p}\,. \tag{5.33}$$

Here we omitted the arguments $(p, q)$ in $\frac{\partial\tau}{\partial v}$. Our goal is to express all terms in (5.33) through $u$ and $x$. First, note that, according to (4.10) and (4.11),

$$\begin{aligned} b^2\cosh q\cosh p &= a^2\coth^2 u - T^2\tanh^2 u\,, \\ b(\cosh q + \cosh p) &= \mu + 4tb^2 = 2a\coth u\,. \end{aligned}$$

Thus

$$\frac{\partial\tau}{\partial v}\left(\frac{i\pi}{2}\right) = b^2\left[-4ti - \frac{b\sinh 2u - 2ai\coth u}{2(a^2\coth^2 u - T^2\tanh^2 u)}\right]. \tag{5.34}$$

Using $u = 4ta + x$, we obtain after some algebra

$$\frac{\partial\tau}{\partial v}\left(\frac{i\pi}{2}\right) = \frac{-\sinh u\sqrt{a^2\coth^2 u - T^2} + i(a\coth u - (u-x)[a\coth^2 u - \frac{T^2\tanh^2 u}{a\coth u}])}{a^2\coth^2 u - T^2\tanh^2 u}\,. \tag{5.35}$$

Multiplying (4.26) by $\coth^2 u$, we obtain $a\coth u = \frac{(\mu+2)u}{2(u+x)} + O(u)$ as $u \to 0$. These formulae show that $R\tau_v(\frac{i\pi}{2}) < 0$ and that the requirement $\zeta \to 0$ as $u \to 0$ implies $\tau_v(\frac{i\pi}{2}) \to 0$. So, we have $x = O(u)$ as $u \to 0$. Moreover, we can calculate the analogue of (5.23),

$$\begin{aligned}\arg\tau_v\left(\frac{i\pi}{2}\right) &= -\frac{\sqrt{\mu+2}[x + \frac{\mu-2}{\mu+2}(u^2-x^2)(u+x)]}{u^2[1 - \frac{\mu-2}{4}\frac{x}{u}(2+\frac{x}{u})]} + O(u)\\ &= -\tan\left(\frac{\sqrt{\mu+2}x}{2u^2} + O(u)\right) + \pi\,.\end{aligned} \tag{5.36}$$

To calculate $\tau_{vvv}(\frac{i\pi}{2})$ we first notice that

$$\begin{aligned}\tau_{vvv}\left(\frac{i\pi}{2}\right) &= (\mu - 2b(\cosh q + \cosh p))i\\ &\quad -\frac{1}{2}ib\left[\frac{\cosh q}{(i+\sinh q)^2} + \frac{\cosh p}{(i+\sinh p)^2}\right]\\ &= \frac{ib}{2}\left[\frac{\cosh q}{(i+\sinh q)^2} + \frac{\cosh p}{(i+\sinh p)^2} - 8tb\right].\end{aligned} \tag{5.37}$$

The sum of the two fractions is

$$\frac{(\cosh q\cosh p - 2)(\cosh q + \cosh p) + 2i\sinh(p+q)}{(i+\sinh q)^2(i+\sinh p)^2}\,. \tag{5.38}$$

Using (4.6) and (4.10), we obtain

$$(i+\sinh q)(i+\sinh p) = -1 + 2i\frac{a}{b} + \frac{a^2-T^2}{b^2} = -\frac{\mu+2}{4} + O(u)\,. \tag{5.39}$$

The imaginary part of the expression in the square brackets of (5.37) is of order $O(u)$. Thus, $\Re\tau_{vvv}(\frac{i\pi}{2}) = O(u)$. Direct calculations, using (5.37)–(5.39), yield

$$\Im\tau_{vvv}\left(\frac{i\pi}{2}\right) = -\frac{32}{(\mu+2)^{3/2}}\,.$$

So, $\arg\tau_{vvv}(\frac{i\pi}{2}) \to -\frac{\pi}{2}$ as $u \to 0$.

Substitution of this expression, together with (5.36) into (5.28), yields

$$\frac{\sqrt{\mu+2}\,x}{2u^2} = \tan\frac{\pi}{10}\,.$$

So, $x = \delta u^2$ as $u \to 0$, where $\delta = (2/\sqrt{\mu+2})\tan\frac{\pi}{10}$. Combining this with (5.31) after some algebra yields the second equation in (5.16). □

## 6 Postbreak Evolution: Higher Genus

In this section we show that conditions (2.19) with $N = 1$ are satisfied on and immediately above the breaking curve $t_0(x)$, i.e., that there exists a function $t_1(x)$ such that $t_0(x) < t_1(x) \le +\infty$ and the region $\{x, t : t_0(x) < t < t_1(x), x \ge 0\}$ is a genus 2 region. We will show that $t_1(x) = +\infty$ for all $x \ge 0$ in the pure radiation case $\mu \ge 2$. In the case $0 \le \mu < 2$, the questions of whether the genus 2 solution breaks and if so how it breaks are still under consideration.

To obtain a genus 2 solution, we need to satisfy the conditions (2.28)–(2.29), equivalent to (2.19), in the case $N = 1$, i.e., when the contour $\gamma_m = \gamma_{m,0} \cup \gamma_{m,1}^+ \cup \gamma_{m,1}^-$ consists of three cuts: $\gamma_{m,0}$ with endpoints $(\bar\alpha_0, \alpha_0)$ as before in the genus 0 case, $\gamma_{m,1}^+$ – a simple contour in the upper half-plane with endpoints $(\alpha_2, \alpha_4)$ that does not intersect $\gamma_{m,0}$ and $\gamma_{m,1}^- = \overline{\gamma_{m,1}^+}$. As shown above, the conditions (2.29) with $N = 0$ break at $t = t_0(x)$, when a point $z_0$ on $\gamma_c$ satisfies $h'(z_0) = 0$. According to Theorem 3.1, the two functions $h(z; x, t_0(x))$, determined by (3.4) for $N = 0$ and $N = 1$, coincide. Here for $N = 1$ we have $\alpha = (\alpha_0, \alpha_2, \alpha_4)$, where $\alpha_2 = \alpha_4 = z_0$ and $\alpha_0$ is equal to $\alpha$ for $N = 0$. The MI system (3.5) and (3.9) with $N = 1$ can be written as four moment and two integral conditions:

$$\frac{1}{\pi i}\int_{\hat\gamma} \frac{\zeta^k f'(\zeta)\,d\zeta}{R(\zeta)} = 0\,,$$

(6.1)

$$\int_{\hat\gamma_{m,c}} h'(z)\,dz = \int_{\hat\gamma_{m,c}} R(z)\,dz \frac{1}{\pi i}\int_{\hat\gamma} \frac{f'(\zeta)\,d\zeta}{(\zeta - z)R(\zeta)} = 0\,,$$

where $k = 0, 1, 2, 3$, the contours $\hat\gamma_m = \hat\gamma_{m,1}$ and $\hat\gamma_c = \hat\gamma_{c,1}$, and

$$R(z) = \sqrt{\prod_{j=0}^{5}(z - \alpha_j)}\,.$$

We show that the solution $\alpha$ to system (6.1) can be continued into some region $P$ containing the breaking curve $t_0(x)$, and that all the required conditions (2.29) are satisfied in $P^+ = \{x, t : t_0(x) < t < t_1(x)\}$, which is the part of $P$ situated above the breaking curve. Then the solution to (2.19) in $P^+$ is given through (3.4), (3.8), and (3.17). That allows us to calculate $q_0(x, t, \varepsilon)$ in $P^+$ by solving the RHP $P^{(\mathrm{mod})}$; see Section 2.7.

The proof that conditions (2.19) with $N = 1$ are satisfied in $P^+$ consists of the following steps: a proof of the Jacobian formula (3.15), which establishes the evolution theorem (Theorem 3.2) in Section 6.1; a proof that the system of MI conditions (6.1), satisfied by $\alpha = (\alpha_0, z_0, z_0)$ on the breaking curve, has a unique continuation into a vicinity of this curve; and that conditions (2.28)–(2.29), equivalent to (2.19), hold above the breaking curve, Section 6.2. Differential equations, satisfied by $\alpha$ in the genus $N$ region, are considered in Section 6.3. The fact that in the case $\mu \geq 2$ the entire region above the breaking curve $t_0(x)$ is a genus 2 region is proven in Section 6.4.

## 6.1 Proof of Theorem 3.3

PROOF: We first prove the theorem for $N = 1$ and then extend the argument for arbitrary $N \in \mathbb{N}$. In the particular case $N = 1$, the Jacobian formula (3.15) becomes

$$\left|\frac{\partial F}{\partial \alpha}\right| = \prod_{j=0}^{2}\left|\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}\right|^2 \prod_{j<l}(\alpha_l - \alpha_j) \cdot \int_{\hat{\gamma}_m}\int_{\hat{\gamma}_c} \frac{(z_1 - z_2)dz_1\, dz_2}{R(z_1)R(z_2)}. \tag{6.2}$$

Using

$$\frac{\partial R^{-1}(z,\alpha)}{\partial \alpha_j} = \frac{1}{2(z-\alpha_j)R(z,\alpha)}$$

and the moment conditions $M_k$, given by (3.5), we obtain

$$\frac{\partial F_k}{\partial \alpha_j} = \frac{1}{2}\frac{1}{2\pi i}\int_{\hat{\gamma}} \frac{\zeta^k f'(\zeta)d\zeta}{(\zeta-\alpha_j)R(\zeta)} = \frac{1}{2}\alpha_j^k \frac{h'(\zeta)}{R(\zeta)}\bigg|_{\zeta=\alpha_j}, \tag{6.3}$$

where $k = 0, 1, 2, 3,\ j = 0, 1, \dots, 5$; all notation was introduced in Section 3.

According to (3.4), the latter ratio is well-defined at $\alpha_j$, $j = 0, 1, \dots, 5$. It will be denoted by $\frac{h'(\alpha_j)}{R(\alpha_j)}$. Calculation of the last two lines $\frac{\partial F_{5,6}}{\partial \alpha}$ of the Jacobian matrix yield

$$\begin{aligned}\frac{\partial F_{5,6}}{\partial \alpha_j} &= \int_{\hat{\gamma}_{m,c}} R(z)dz \frac{1}{4\pi i}\int_{\hat{\gamma}}\left[\frac{1}{\zeta - a_j} - \frac{1}{z-a_j}\right]\frac{f'(\zeta)d\zeta}{(\zeta - z)R(\zeta)} \\ &= -\frac{1}{2}\frac{h'(\alpha_j)}{R(\alpha_j)}\int_{\hat{\gamma}_{m,c}}\frac{R(z)dz}{(z-\alpha_j)}.\end{aligned} \tag{6.4}$$

Thus,

$$\left|\frac{\partial F}{\partial \alpha}\right| = \prod_{j=0}^{2}\left|\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}\right|^2 \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_5 \\ \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_5^2 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_5^3 \\ \int_{\hat\gamma_m}\frac{R(z)dz}{(z-\alpha_0)} & \int_{\hat\gamma_m}\frac{R(z)dz}{(z-\alpha_1)} & \cdots & \int_{\hat\gamma_m}\frac{R(z)dz}{(z-\alpha_5)} \\ \int_{\hat\gamma_c}\frac{R(z)dz}{(z-\alpha_0)} & \int_{\hat\gamma_c}\frac{R(z)dz}{(z-\alpha_1)} & \cdots & \int_{\hat\gamma_c}\frac{R(z)dz}{(z-\alpha_5)} \end{pmatrix}$$

$$(6.5)\qquad = \prod_{j=0}^{2}\left|\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}\right|^2 \int_{\hat\gamma_m}\int_{\hat\gamma_c}\frac{dz_1dz_2}{R(z_1)R(z_2)}$$

$$\times \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_5 \\ \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_5^2 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_5^3 \\ \frac{\prod_{l=0}^5(z_1-\alpha_l)}{(z_1-\alpha_0)} & \frac{\prod_{l=0}^5(z_1-\alpha_l)}{(z_1-\alpha_1)} & \cdots & \frac{\prod_{l=0}^5(z_1-\alpha_l)}{(z_1-\alpha_5)} \\ \frac{\prod_{l=0}^5(z_2-\alpha_l)}{(z_2-\alpha_0)} & \frac{\prod_{l=0}^5(z_2-\alpha_l)}{(z_2-\alpha_1)} & \cdots & \frac{\prod_{l=0}^5(z_2-\alpha_l)}{(z_2-\alpha_5)} \end{pmatrix}.$$

To compute the latter determinant, note that

$$(6.6)\qquad \prod_{l=0}^{5}(z-\alpha_l) = z^6 + \sum_{l=0}^{5}\phi_{6-l}z^l\,,$$

where $\phi_l$ are the standard symmetric functions of $\alpha$'s, for example, $\phi_1 = -\sum_j \alpha_j$, $\phi_2 = \sum_{k<j}\alpha_k\alpha_j$, etc. The next observation is that the coefficients $\phi_{r,n}$ of the polynomial

$$(6.7)\qquad \frac{\prod_{l=0}^5(z-\alpha_l)}{(z-\alpha_s)} = z^5 + \sum_{l=1}^{5}\phi_{6-l,s}z^{l-1}\,,$$

where $s = 0, 1, \ldots 5$, are themselves polynomials in $\alpha_s$ of order $r$ with coefficients expressed in terms of $\phi_1, \ldots, \phi_r$. For example, $\phi_{1,s} = -\phi_1 + \alpha_s$, $\phi_{2,s} = \phi_2 - \alpha_s\phi_1 + \alpha_s^2$, etc. Then, by adding proper linear combinations of the first four rows of the determinant to the last two rows, we kill all but the last two terms in the polynomials $\phi_{4,s}$ and $\phi_{5,s}$. Since

$$\phi_{5,s} = -\frac{\prod_{l=0}^5\alpha_l}{\alpha_s}\,,\qquad \phi_{4,s} = \frac{\prod_{l=0}^5\alpha_l}{\alpha_s}\left[\sum_{l=0}^{5}\frac{1}{\alpha_l} - \frac{1}{\alpha_s}\right],$$

the determinant from (6.5) becomes

$$\det\begin{pmatrix} \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_5^2 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_5^3 \\ \alpha_0^4 & \alpha_1^4 & \cdots & \alpha_5^4 \\ \alpha_0^5 & \alpha_1^5 & \cdots & \alpha_5^5 \\ [\alpha_0\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_1 - \alpha_0 & [\alpha_1\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_1 - \alpha_1 & \cdots & [\alpha_5\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_1 - \alpha_5 \\ [\alpha_0\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_2 - \alpha_0 & [\alpha_1\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_2 - \alpha_1 & \cdots & [\alpha_5\sum_{l=0}^5 \frac{1}{\alpha_l} - 1]z_2 - \alpha_5 \end{pmatrix} = (z_1 - z_2)\prod_{j<l}(\alpha_l - \alpha_j)\,, \tag{6.8}$$

where the last expression was obtained after taking proper linear combinations of the last two rows and computing the corresponding Vandermonde determinant. Combining (6.5) and (6.8), we obtain (6.2). The proof for $N = 1$ is completed.

In the case of a general $N$, we observe that the "Vandermonde part" of the first determinant in (6.5) consists of the first $2N + 2$ rows, and the "integrals part" consists of the last $2N$ rows. Taking the integration outside, we obtain the determinant similar to the second determinant in (6.5), where the $(2N + 2 + m)^{\text{th}}$ row

$$L_m = (p_0(z_m), p_1(z_m), \ldots, p_{4N+1}(z_m))\,, \tag{6.9}$$

$m = 1, \ldots, 2N$, consists of the polynomials

$$p_s(z_m) = z_m^{4N+1} + \sum_{l=1}^{4N+1} \phi_{4N+2-l,s} z_m^{l-1}\,. \tag{6.10}$$

Here $\phi_{l,s} = \alpha_s^l + \sum_{k=0}^{l-1} c_k \alpha_s^k$ and $c_k$ are symmetrical polynomials in $\alpha_j$, $j = 0, 1, \ldots, 4N + 1$. Because of this structure of $\phi_{l,s}$, one can eliminate all terms of degree $z_m^{2N}$ and higher in all the polynomials (6.10) by taking proper linear combinations with the first $2N + 2$ rows of the Vandermonde part of the determinant. Thus, we can assume that the polynomials $p_s(z_m)$ in (6.9) are

$$p_s(z_m) = \alpha_s^{2N+2} z_m^{2N-1} + \sum_{l=1}^{2N-1} \tilde{\phi}_{4N+2-l,s} z_m^{l-1}\,, \tag{6.11}$$

where $\tilde{\phi}_{l,s}$ are symmetric polynomials in $\alpha_j$, $j = 0, 1, \ldots, 4N + 1$ .

To obtain a $(4N + 2) \times (4N + 2)$ Vandermonde determinant, we proceed with the Gaussian elimination process within the rows $L_m$. In particular, subtracting $L_{2N}$ from the rest of $L_m$ and factoring out $z_m - z_{2N}$ in each row, we obtain new rows of the form

$$L_m^{(1)} = (z_m - z_{2N})\big(p_0^{(1)}, p_1^{(1)}, \ldots, p_{4N+1}^{(1)}\big)\,, \tag{6.12}$$

where

$$p_s^{(1)}(z_m, z_{2N}) = \alpha_s^{2N+2}\left(z_m^{2N-1} - z_{2N}^{2N-1}\right) + \sum_{l=2}^{2N-1} \phi_{4N+2-l,s}\left(z_m^{l-1} - z_{2N}^{l-1}\right). \tag{6.13}$$

We continue the process by subtracting row $L_{2N-1}^{(1)}$ from the previous rows $L_m^{(1)}$, $m = 1, \dots, 2N-2$, and factoring out $z_m - z_{2N-1}$. Eventually the first row of the integrals part will become $(\alpha_0^{2N+2}, \alpha_1^{2N+2}, \dots, \alpha_{2N+1}^{2N+2})$ multiplied by $\prod_{l=2}^{2N}(z_1 - z_l)$. Using now the inverse part of the Gauss method, we found that the $m^{\text{th}}$ row of the integrals part is $(\alpha_0^{2N+1+m}, \alpha_1^{2N+1+m}, \dots, \alpha_{2N+1}^{2N+1+m})$ multiplied by $\prod_{l=m+1}^{2N}(z_m - z_l)$. Thus, we obtain the required expression for the Jacobian

$$\left|\frac{\partial F}{\partial \alpha}\right| = \prod_{j=0}^{2N}\left|\frac{h'(\alpha_{2j})}{2R(\alpha_{2j})}\right|^2 \prod_{j<l}(\alpha_l - \alpha_j) \int_{\hat{\gamma}_{m,1}}\int_{\hat{\gamma}_{c,1}} \cdots \int_{\hat{\gamma}_{m,N}}\int_{\hat{\gamma}_{c,N}} \prod_{j<l}^{2N}(z_j - z_l)\prod_{k=1}^{2N}\frac{dz_k}{R(z_k)}, \tag{6.14}$$

where the latter integral can be written as the determinant (3.16). □

COROLLARY 6.1 *For any nondegenerate* $\alpha$*, the Jacobian* (3.15)–(3.16) *is different from* 0*.*

The proof of the corollary follows from the definition of degeneracy (see Section 3) and properties of holomorphic differentials of the Riemann surface $\mathcal{R}$ of $R$.

Effectively, the corollary says that for our $f$ the Jacobian is nonzero if all $\alpha$ are distinct, since for all such $\alpha$ we have $h'(z) \sim O(z - a_j)^{1/2}$ when $z$ approaches $\alpha_j$. Moreover, we have seen in Section 4.6 that $h'(z)/(2R(z)) \neq 0$ even if two or three branch points $\alpha_{2j}$ collide at some point $z$ in the upper half-plane.

## 6.2 Transition from Genus 0 to Genus 2

According to Sections 4.5 and 4.6, the solution to (2.19) with $N = 0$ was established in the region $x \geq 0$, $t \in [0, t_0(x))$. On the breaking curve itself, i.e., when $t = t_0(x)$, $x > 0$, the inequality on the single complementary arc $\gamma_c^+$ in (2.19) breaks down, but all other conditions still hold. Thus, according to the degeneracy theorem (Theorem 3.1), we can obtain a solution $\alpha = (\alpha_0, \alpha_2, \alpha_4)$, where $\alpha_2 = \alpha_4$, to the system MI (6.1) when $(x, t)$ is on the breaking curve. Our next goal is to show that this solution can be continued into a vicinity of $(x, t)$. The solution on the breaking curve is degenerate, leading to the Jacobian $|\frac{\partial F}{\partial \alpha}| = 0$. We overcome this difficulty by using a smooth change of variables that transforms the Jacobian (6.2) to one that is different from zero. We then show that the function $g$ and the constants $W$ and $\Omega$, determined through $\alpha$ by (3.3) and (3.8), satisfy conditions (2.19) with $N = 1$ in some region $\{(x, t) : t_0(x) < t < t_1(x)\}$ above the breaking curve.

Given some $(x,t)$ and $\alpha$ satisfying (6.1), the RHP (3.1) uniquely determines the function $g'(z)$ by (3.3), which, in turn, uniquely determines $g$ and $h$. Let us denote these objects by subscript 2, since $N=1$ corresponds to genus $2N=2$. Similarly, we will use the subscript 0 in the genus 0 case, i.e., when there is only one interval $\gamma_0^+$ with endpoints $\frac{\mu}{2}$ and $\alpha_0$ in the RHP (3.1), where $\alpha_0$ was calculated in Section 4.1 and $\Re\alpha_0 \geq 0$. According to the degeneracy theorem (Theorem 3.1), the system of MI conditions (6.1) coincides with systems (4.2) and (4.88), and $h_0(z;x,t) \equiv h_2(z;x,t)$ when $(x,t)$, $x>0$, is on the breaking curve. It also follows from (2.19) and (3.8) that $\Omega_1 = 0$ and $W_1 = \Re h(\alpha_2)$. The case $x=0$ and $t = \frac{1}{2\sqrt{\mu+2}}$ corresponds to $\alpha_0 = \alpha_2 = \alpha_4 = i\sqrt{\mu+2}$.

Let us now consider the behavior of the Jacobian (6.2) near a point $(x, t_0(x))$, $x \neq 0$, on the breaking curve, i.e., in the limit $2\delta = \alpha_2 - \alpha_4 \to 0$ and, correspondingly, $2\bar{\delta} = \alpha_3 - \alpha_5 \to 0$, when no other pairs of $\alpha$'s approach each other.

LEMMA 6.2 *There exists a constant $A \neq 0$ such that*

$$\left|\frac{\partial F}{\partial \alpha}\right| = A|\delta|^2 \ln|\delta|[1+o(\delta)] \tag{6.15}$$

*as $\delta \to 0$.*

PROOF: Note that there are exactly three zero level curves of $\Im h$ emanating from any $\alpha_j$. That means $h'(\alpha_j)/R(\alpha_j) \in \mathbb{C}^*$, i.e., is finite and nonzero. The counting of zero level curves of $\Im h$ shows that the latter fact would not change if two or more points $\alpha_j$ coincide. Thus, in the limit $\delta \to 0$, the first product in (6.2) approaches a certain nonzero value, whereas the second product behaves like a nonzero constant times $|\delta|^2$. To understand the behavior of the integral factor in (6.2), note that

$$\begin{aligned} R^{-1}(z) &= ((z-L)^2-\delta^2)^{-\frac{1}{2}}((z-\bar{L})^2-\bar{\delta}^2)^{-\frac{1}{2}} R_0^{-1}(z) \\ &= \frac{1}{2i\Im L}\left[\frac{1}{z-L} - \frac{1}{z-\bar{L}}\right] R_0^{-1}(z)[1+O(|\delta|^2)]\,, \end{aligned} \tag{6.16}$$

where $2L = \alpha_2 + \alpha_4$ and $R_0 = \sqrt{(z-\alpha_0)(z-\alpha_1)}$. Thus, changing the order of integration in the double integral in (6.2), we evaluate this integral as

$$\begin{aligned} D &= \int_{\hat{\gamma}_c} \frac{dz_2}{R(z_2)} \int_{\hat{\gamma}_m} \frac{(z_1-z_2)dz_1}{R(z_1)} \\ &= \frac{\pi}{\Im L}\int_{\hat{\gamma}_c}\left[\frac{L-z_2}{R_0(L)R(z_2)} - \frac{\bar{L}-z_2}{R_0(\bar{L})R(z_2)}\right]dz_2 + O(|\delta|^2)\,. \end{aligned} \tag{6.17}$$

Note that the first term of the integrand has a singularity near the point $z_2 = \bar{L}$ as $\delta \to 0$, whereas the second term has a singularity near the point $z_2 = L$. To evaluate the leading-order behavior of these terms as $\delta \to 0$, we need to integrate only in the vicinities of the corresponding points $L$ and $\bar{L}$, where we can assume

$R(z) \approx \sqrt{(z-L)^2-\delta^2}(L-\bar{L})R_0(L)$ and $R(z) \approx \sqrt{(z-\bar{L})^2-\bar{\delta}^2}(\bar{L}-L)R_0(\bar{L})$, respectively. Substituting these expressions into (6.17), we obtain

$$D \to \frac{\pi}{\Im L|R_0(L)|^2}\Bigg[\int_{\hat{\gamma}_c^+} \frac{dz}{\sqrt{(z-L)^2-\delta^2}} - \int_{\hat{\gamma}_c^-} \frac{dz}{\sqrt{(z-\bar{L})^2-\bar{\delta}^2}}\Bigg]$$
(6.18)
$$= \frac{2\pi}{\Im L|R_0(L)|^2}\Re\Bigg[\int_{\hat{\gamma}_c^+} \frac{dz}{\sqrt{(z-L)^2-\delta^2}}\Bigg] \to -\frac{2\pi}{\Im L|R_0(L)|^2}\ln|\delta|$$

as $\delta \to 0$. Here $\hat{\gamma}_c^{\pm}$ denote small pieces of the contour $\hat{\gamma}_c$ close to the points $L$ and $\bar{L}$, respectively. Now we see that (6.15) follows from (6.18). □

By making a proper change of variables, we can reduce the Jacobian $|\frac{\partial F}{\partial \alpha}|$ to a nonzero constant in the limit $\delta \to 0$. However, we simultaneously need to show that the entries of the corresponding Jacobian matrix are bounded as $\delta \to 0$. To find the required change of variables, we need the following statement:

PROPOSITION 6.3 *There exists* $\phi \in [0, 2\pi)$ *such that* $\lim_{\delta \to 0} \arg \delta = \phi$.

PROOF: Let us write the first integral condition $I_1$ of (6.1) as

$$\frac{1}{\pi i}\int_{\hat{\gamma}} \frac{f'(\zeta)d\zeta}{R(\zeta)} \int_{\hat{\gamma}_m} \frac{A_0(z)+A_1(z,\delta)}{\zeta - z}\,dz = 0\,, \tag{6.19}$$

where $R(z) = A_0(z) + A_1(z,\delta)$ with $A_1 \sim O(|\delta|^2)$. Using the same arguments as in (6.16), we obtain $A_0(z) = (z-L)(z-\bar{L})R_0(z)$ and $A_1(z,\delta) = -\frac{1}{2}[\delta^2\frac{z-\bar{L}}{z-L} + \bar{\delta}^2\frac{z-L}{z-\bar{L}}]R_0(z) + o(\delta^2)$. Using residues, we compute the inner integral as $2\pi\Im L[\delta^2\frac{R_0(L)}{\zeta-L} - \bar{\delta}^2\frac{R_0(\bar{L})}{\zeta-\bar{L}}]$. Thus (6.19) becomes

$$2\pi\Im L\Bigg[\delta^2\frac{h'(L)R_0(L)}{R(L)} - \bar{\delta}^2\frac{h'(\bar{L})R_0(\bar{L})}{R(\bar{L})}\Bigg] + o(\delta^2) = 4\pi i\Im L\Im\Bigg[\delta^2\frac{h'(L)R_0(L)}{R(L)}\Bigg] + o(\delta^2) = 0\,. \tag{6.20}$$

Hence, in the limit $\delta \to 0$, we obtain $\arg\delta^2 = -\theta + \pi n$, where $\theta = \arg\frac{h'(L)R_0(L)}{R(L)}$ and $n \in \mathbb{Z}$. Thus, the statement is proven. □

Note that $n$ can change only if $\delta^2\frac{h'(L)R_0(L)}{R(L)}$ passes through 0, i.e., only if $\delta = 0$. Thus, the change of $\arg\delta$ cannot occur anywhere in a vicinity of a breaking point but at the point itself.

THEOREM 6.4 *A solution* $\alpha$ *to* (6.1) *on the breaking curve can be uniquely extended into some region containing the breaking curve.*

PROOF: On the breaking curve the system has a solution, as was shown above. The change of variables $\alpha \mapsto \tilde{\alpha}$, where $\tilde{\alpha} = \{\alpha_0, \alpha_1, L, \bar{L}, \delta, \bar{\delta}\}$, changes the Jacobian (6.15) by a constant nonzero factor. On the other hand, all columns of the matrix $\frac{\partial F}{\partial \tilde{\alpha}}$ are well-defined on the breaking curve except the partial derivatives

$$\frac{\partial F}{\partial \delta} = \frac{\partial F}{\partial \alpha_2} - \frac{\partial F}{\partial \alpha_4}, \qquad \frac{\partial F}{\partial \delta} = \frac{\partial F}{\partial \alpha_2} - \frac{\partial F}{\partial \alpha_4}, \tag{6.21}$$

where

$$\frac{\partial F}{\partial \alpha_j} = \frac{1}{2}\frac{h'(\alpha_j)}{R(\alpha_j)} C_j$$

with $C_j$ denoting the $j^{\text{th}}$ column in (6.5). It is easy to see that

$$\frac{h'(\alpha_2)}{R(\alpha_2)} - \frac{h'(\alpha_4)}{R(\alpha_4)} \sim O(\delta) \quad \text{as } \delta \to 0 .$$

The first five entries of $C_2$ through $C_4$ possess the same property, whereas for the last entry, similar to (6.18), we obtain

$$\begin{aligned} \int_{\hat{\gamma}_c} \left[ \frac{1}{z-\alpha_2} - \frac{1}{z-\alpha_4} \right] R(z)dz &= 2\delta \int_{\hat{\gamma}_c} \frac{R(z)dz}{(z-\alpha_2)(z-\alpha_4)} \\ &\approx 4i R_0(L) \Im(L) \delta \ln|\delta| . \end{aligned} \tag{6.22}$$

The corresponding expression for $\frac{\partial F}{\partial \bar{\delta}}$ is $4i R_0(\bar{L})\Im(L)\bar{\delta} \ln|\delta|$. Thus,

$$\begin{aligned} \frac{\partial F}{\partial \delta} &= 2i \frac{h'(L)}{R(L)} R_0(L)\Im(L)\delta \ln|\delta| + O(\delta) , \\ \frac{\partial F}{\partial \bar{\delta}} &= 2i \frac{h'(\bar{L})}{R(\bar{L})} R_0(\bar{L})\Im(L)\bar{\delta} \ln|\delta| + O(\bar{d}) . \end{aligned} \tag{6.23}$$

Introducing new variables $\sigma = \delta^2$ and $\bar{\sigma} = \bar{\delta}^2$, we obtain that the last entries of $\frac{\partial F}{\partial \sigma}$ and of $\frac{\partial F}{\partial \bar{\sigma}}$ become

$$\begin{aligned} \frac{\partial F_6}{\partial \sigma} &= \frac{1}{2} i \frac{h'(L)}{R(L)} R_0(L)\Im(L) \ln|\sigma| + O(1) , \\ \frac{\partial F_6}{\partial \bar{\sigma}} &= \frac{1}{2} i \frac{h'(\bar{L})}{R(\bar{L})} R_0(\bar{L})\Im(L) \ln|\sigma| + O(1) . \end{aligned} \tag{6.24}$$

All other entries of these columns do not have singularities in $\sigma$.

Introducing now new variables $s_1$ and $s_2$ by

$$\sigma = e^{-i\theta} \frac{s_1}{\ln|s_1|} - e^{-i\theta} s_2 , \quad \bar{\sigma} = e^{i\theta} s_2 , \tag{6.25}$$

where $\theta$ was defined earlier as $\arg[\frac{h'(L)}{R(L)} R_0(L)]$, and using (6.24), we obtain

$$\begin{aligned} \frac{\partial F_6}{\partial s_2} &= -e^{-i\theta}\frac{\partial F_6}{\partial \sigma} + e^{i\theta}\frac{\partial F_6}{\partial \bar\sigma} = O(1)\,, \\ \frac{\partial F_6}{\partial s_1} &= e^{-i\theta}\frac{1}{\ln|s_1|}\frac{\partial F_6}{\partial \sigma}\,. \end{aligned} \tag{6.26}$$

Note that $\frac{s_1}{\ln|s_1|} = e^{i\theta}\sigma + e^{-i\theta}\bar\sigma = \pm 2|\sigma|$, where the positive or the negative sign holds if $\arg\sigma = -\theta$ or if $\arg\sigma = -\theta + \pi$, respectively. Therefore, $\ln|\sigma| \sim \ln|s_1|$ as $\sigma \to 0$. This implies that the second equation in (6.26) is $O(1)$. Thus, the Jacobian matrix in the variables $\alpha_0, \alpha_1, L, \bar L, s_1$, and $s_2$ is bounded.

According to (6.15), the Jacobian $J_1$ in the variables $\alpha_0, \alpha_1, L, \bar L, \sigma$, and $\bar\sigma$ has the leading term $O(\ln|s_1|)$. Since

$$\left|\frac{\partial(\sigma,\bar\sigma)}{\partial(s_1,s_2)}\right| = \det\begin{pmatrix} e^{-i\theta}\frac{1}{\ln|s_1|}(1-\frac{1}{\ln|s_1|}) & -e^{-i\theta} \\ 0 & e^{i\theta}\end{pmatrix}\frac{1}{\ln|s_1|}\,, \tag{6.27}$$

the Jacobian $J_2$ in the variables $\alpha_0, \alpha_1, L, \bar L, s_1$, and $s_2$ is

$$J_2 = J_1\left|\frac{\partial(\sigma,\bar\sigma)}{\partial(s_1,s_2)}\right| = O(1) \tag{6.28}$$

as $\sigma \to 0$. We can now apply the implicit function theorem to show the existence of a unique solution to (6.1) in the vicinity of $x_0$. Thus, the proof is complete. □

LEMMA 6.5 *The variable* $\delta = \alpha_2 - \alpha_4$ *changes its argument by* $\frac{\pi}{2}$ *when* $x$ *is crossing the breaking curve through a point* $(x_0, t_0)$*,* $x_0 > 0$.

PROOF: Let us first prove the statement for fixed $t_0$. If $\hat\alpha$ denotes the variables $\{\alpha_0, \alpha_1, L, \bar L, s_1, s_2\}$, then, as it follows from the proof of the previous theorem, $\hat\alpha_x = -(F_{\hat\alpha})^{-1}F_x$ is bounded near the point $(x_0, t_0)$. We want to show that at least one of the values $(s_1)_x = (\hat\alpha_5)_x$ and $(s_2)_x = (\hat\alpha_6)_x$ is different from zero. In order to do so, we calculate

$$\frac{\partial F}{\partial\hat\alpha} = \frac{\partial F}{\partial\alpha}\left(\frac{\partial\hat\alpha}{\partial\alpha}\right)^{-1}. \tag{6.29}$$

Direct calculation shows that the $6\times 6$ matrix $\frac{\partial\alpha}{\partial\hat\alpha}$, written as a block matrix of $2\times 2$ blocks, is

$$\frac{\partial\alpha}{\partial\hat\alpha} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & M \\ 0 & I & -M\end{pmatrix} \quad\text{where}\quad M = \frac{1}{2}\begin{pmatrix} -\frac{e^{-i\theta}}{\sqrt{\sigma}\ln|\sigma|} & \frac{e^{-i\theta}}{\sqrt{\sigma}} \\ 0 & \frac{e^{i\theta}}{\sqrt{\bar\sigma}}\end{pmatrix}. \tag{6.30}$$

Then the linear system of equations for $\hat\alpha_x$, written in block matrix form, becomes

$$\begin{pmatrix} F_{11} & F_{12}+F_{13} & (F_{12}-F_{13})M \\ F_{21} & F_{22}+F_{23} & (F_{22}-F_{23})M \\ F_{31} & F_{32}+F_{33} & (F_{32}-F_{33})M\end{pmatrix}\hat\alpha_x = -F_x\,, \tag{6.31}$$

where $\frac{\partial F}{\partial\hat\alpha} = \{F_{ij}\}$, $i, j = 1, 2, 3$.

Let us now assume that $(\hat\alpha_5)_x = (\hat\alpha_6)_x = 0$ and show that this assumption leads to a contradiction. Indeed, taking into account (6.5) and considering the limit $\delta \to 0$, the system (6.31) becomes the overdetermined system of five equations and four unknowns

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & L & \bar L \\ \alpha_0^2 & \alpha_1^2 & L^2 & \bar L^2 \\ \alpha_0^3 & \alpha_1^3 & L^3 & \bar L^3 \\ \int_{\hat\gamma_c} \frac{R(z)dz}{z-\alpha_0} & \int_{\hat\gamma_c} \frac{R(z)dz}{z-\alpha_1} & \int_{\hat\gamma_c} \frac{R(z)dz}{z-L} & \int_{\hat\gamma_c} \frac{R(z)dz}{z-\bar L} \end{pmatrix} \beta = -F_x\,, \tag{6.32}$$

where

$$\begin{aligned} \beta &= \mathrm{Col}(\beta_1, \beta_2, \beta_3, \beta_4) \\ &= \mathrm{Col}\left( \frac{h'(\alpha_0)}{2R(\alpha_0)}(\alpha_0)_x, \frac{h'(\alpha_1)}{2R(\alpha_1)}(\alpha_1)_x, \frac{h'(L)}{2R(L)}L_x, \frac{h'(\bar L)}{2R(\bar L)}\bar L_x \right). \end{aligned}$$

Here we took into account the fact that all the integrals $\int_{\hat\gamma_m}$ in the fifth row of (6.5) approach 0 as $\delta \to 0$, so that the fifth equation in the system (6.31) becomes trivial in this limit.

Using the same arguments as in the proof of Theorem 6.1 and the fact that $R(z) = (z-L)(z-\bar L)R_0(z)$, and subtracting a linear combination of the first two equations from the last equation of (6.32), we reduce the last row of the matrix in (6.32) to

$$\int_{\hat\gamma_c} \mathrm{Col}\big(\alpha_0^2(z-\phi)+\alpha_0^3, \alpha_1^2(z-\phi)+\alpha_1^3, L^2(z-\phi)+L^3, \bar L^2(z-\phi)+\bar L^3\big)\frac{dz}{R_0(z)}\,, \tag{6.33}$$

where $\phi = 2\Re(\alpha_0 + L)$. Observing that this row is a linear combination of the third and fourth rows of (6.32), we can rewrite the last equation of (6.32) after some algebra as

$$\Im \int_{\alpha_0}^{L} \frac{(z-\alpha_0)dz}{R_0(z)} = 0\,. \tag{6.34}$$

This equation is equivalent to $\Im R_0(L) = 0$, which contradicts (4.46) and Lemma 4.5. According to the arguments in the proof of Theorem 6.4, $s_2 = o(s_1)$ as $\delta \to 0$. Thus, we obtain $(s_1)_x \neq 0$, whereas $(s_2)_x = 0$. This implies that

$$\delta = O\left(\frac{x-x_0}{\ln|x-x_0|}\right)^{\frac12} \quad \text{as } x \to x_0\,.$$

It is now clear that $\arg\delta$ changes by $\frac{\pi}{2}$ as $x$ passes through $x_0$. According to (6.20), the only way $\lim_{\delta\to 0}\arg\delta$ can experience a change is when the point $(x,t)$ crosses the breaking curve. Therefore, the change of $\arg\delta$ by $\frac{\pi}{2}$ occurs regardless of how $(x,t)$ crosses the breaking curve. The proof is complete. □

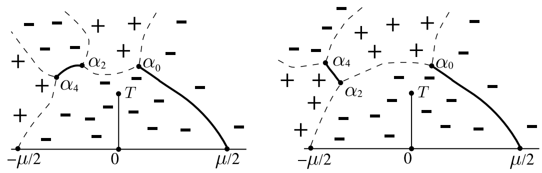


FIGURE 6.1. Possible topology of the level curves of $\Im h$.

THEOREM 6.6 *Conditions* (2.19) *with* $N = 1$ *are satisfied in some region above the breaking curve.*

PROOF: To establish the existence of $g$ in the genus 2 region, we need to show that in addition to existence of the three main arcs with endpoints $\alpha_j$, $j = 0, 1, \dots, 5$, symmetrical with respect to the real axis, we also have the right topology of the level curves of $\Im h(z,x,t)$, i.e., that $\Im h(z,x,t) < 0$ on both sides of the cut from $\alpha_2$ to $\alpha_4$. This condition is necessary for the required inequalities on $h$ to be satisfied. Considering branches of the zero level curves of $\Im h(z,x,t)$ in the upper half-plane, we observe that there are three branches of this curve going to infinity, two branches emanating from each of the points $\alpha_0$, $\alpha_2$, and $\alpha_4$, and one branch emanating from $-\frac{\mu}{2}$. One additional branch connects the points $\alpha_2$ and $\alpha_4$ and the points $\frac{\mu}{2}$ and $\alpha_0$. The only two possible topological portraits of zero level curves of $\Im h(z,x,t)$, characterized by the sign of $\Im h(z,x,t)$ on both sides of the cut $\alpha_2$ and $\alpha_4$, are shown in Figure 6.1.

In the prebreak region the genus 0 asymptotics for the NLS (1.1)–(1.2) is valid, so that the genus 2 asymptotics cannot be valid there. To be more precise, the corresponding leading-order terms are different, and the error estimates from the next section complete the argument. Thus, the sign of $\Im h(z,x,t)$ on both sides of the cut from $\alpha_2$ to $\alpha_4$ before the break is positive. To show the transition to genus 2 asymptotics after crossing the breaking curve, we need to show the change of sign of $\Im h(z,x,t)$ after the break.

Indeed, the function $\frac{h'(z)}{R(z)}$ is analytic in a vicinity of the main arc $\gamma_m^+$ and has limits as $z$ approaches the endpoint $\alpha_2$ or $\alpha_4$ of $\gamma_m^+$. Moreover, these limits continuously depend on $\alpha_2$ and $\alpha_4$, even if they coincide with each other. Thus, in a small vicinity (say, of order $\delta^2$, where $\alpha_2 - \alpha_4 = 2\delta$) of $\alpha_2$, we have

$$h'(z) = h_0(z-\alpha_2)^{\frac{1}{2}} + o(z-\alpha_2)^{\frac{1}{2}} = K\sqrt{\delta}(z-\alpha_2)^{\frac{1}{2}}\,,$$

where $K \in \mathbb{C}^*$. Then

$$h(z) = \frac{2}{3}K\sqrt{\delta}(z-\alpha_2)^{\frac{3}{2}}(1+o(1))\,.$$

Choosing $\arg(z-\alpha_2) = \arg\delta$, we obtain

$$\arg h(z) = 2\arg\delta + \arg K\,. \tag{6.35}$$

Since, according to Lemma 6.5, $\arg\delta$ changes by $\frac{\pi}{2}$ after crossing the breaking curve, we conclude that $\Im h$ changes its sign on the ray $\arg(z-\alpha_2) = \arg\delta$ and

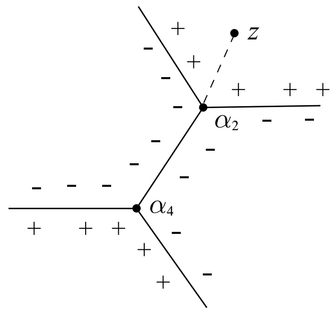


FIGURE 6.2. Change of sign after the break.

small $z - \alpha_2$ after crossing the breaking curve, providing $\Im h \neq 0$. But the last condition is guaranteed since $z$ lies between zero level curves of $\Im h$; see Figure 6.2.

So, $\Im h$ also changes sign on both sides of the main arc $\gamma_m^+$ after crossing the breaking curve. The proof is complete. $\square$

## 6.3 Differential Equations for $\alpha$

In this subsection we derive differential equations for $\alpha$, which are used to extend the solution to the system $F(\alpha, x, t) = 0$ of Theorem 6.4 from a vicinity of the breaking curve to the region $Q$ of the $x, t$-plane where all $\alpha$'s are finite and stay away from the real axis. The breaking curve itself cuts $Q$ into two parts, $Q = Q^+ \cup Q^-$, which lie above or below the breaking curve, respectively. It will be shown later that in the case $\mu \geq 2$ the statement of Theorem 6.6 can be extended from a vicinity of the breaking curve onto the whole region $Q^+$, i.e., that conditions (2.19) with $N = 1$ are satisfied in $Q^+$. That means $Q^+ \subseteq P^+$.

The system of equations $F(\alpha, x, t) = 0$, given by (6.1), gives rise to the sets of ordinary differential equations

$$\alpha_x = -(F_\alpha)^{-1} F_x \tag{6.36}$$

and

$$\alpha_t = -(F_\alpha)^{-1} F_t \,, \tag{6.37}$$

where $F_\alpha = \frac{\partial F}{\partial \alpha}$ and the partial derivatives

$$\begin{aligned} F_x &= \operatorname{Col}\left(0, 0, 1, \frac{1}{2}\sum_{j=0}^{5} \alpha_j, 0, 0\right), \\ F_t &= 4\operatorname{Col}\left(0, 1, \frac{1}{2}\sum_{j=0}^{5} \alpha_j, \frac{3}{8}\sum_{j=0}^{5} \alpha_j^2 + \frac{1}{4}\sum_{i<j} \alpha_i\alpha_j, 0\right), \end{aligned} \tag{6.38}$$

are computed using the residue formula at infinity. According to Cramer's rule,

$$(\alpha_j)_x = -\frac{|F_{jx}|}{|F_\alpha|} \tag{6.39}$$

and

$$(\alpha_j)_t = -\frac{|F_{jt}|}{|F_\alpha|}\,, \tag{6.40}$$

where $\alpha_j$, $j = 0, 1, \dots, 5$, denotes the $j^{\text{th}}$ component of vector $\alpha$, and $|F_{jx}|$ and $|F_{jt}|$ denote the determinant $|F_\alpha|$, given by (6.14), with the $j+1^{\text{th}}$ column replaced by $F_x$ or $F_t$, respectively.

LEMMA 6.7 *The solution $\alpha$ to* (6.1)*, established in Theorem* 6.4*, has a unique extension onto $Q$.*

The proof of the statement follows the existence and uniqueness theorem, since (6.36) and (6.37) are autonomous systems of differential equations, where the right-hand sides are $C^1$ vector functions in $Q$.

In the rest of this section we derive the equations for $\alpha_x$ and $\alpha_t$. Let us first focus on the equation (6.39). Since all but two entries of $F_x$ are zeros, we get

$$|F_{jx}| = (-1)^{j+4} \prod_{k \neq j} \frac{h'(\alpha_k)}{2R(\alpha_k)} \left( M_{3,j} - \frac{1}{2} \sum_{j=0}^{5} \alpha_j M_{4,j} \right), \tag{6.41}$$

where $M_{l,j}$ denotes the $(l, j)$ minor of the matrix in (6.5). In particular,

$$M_{4,6} = \tag{6.42}$$

$$\int_{\hat\gamma_m} \int_{\hat\gamma_c} \frac{dz_1 dz_2}{R_5(z_1) R_5(z_2)} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_4 \\ \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_4^2 \\ \frac{\prod_{l=0}^4 (z_1-\alpha_l)}{(z_1-\alpha_0)} & \frac{\prod_{l=0}^4 (z_1-\alpha_l)}{(z_1-\alpha_1)} & \cdots & \frac{\prod_{l=0}^4 (z_1-\alpha_l)}{(z_1-\alpha_4)} \\ \frac{\prod_{l=0}^4 (z_2-\alpha_l)}{(z_2-\alpha_0)} & \frac{\prod_{l=0}^4 (z_2-\alpha_l)}{(z_2-\alpha_1)} & \cdots & \frac{\prod_{l=0}^4 (z_2-\alpha_l)}{(z_2-\alpha_4)} \end{pmatrix},$$

where

$$R_j(z) = \frac{\sqrt{\prod_{l=0}^5 (z-\alpha_l)}}{(z-\alpha_j)}\,.$$

Note that if

$$\prod_{l=0}^{4} (z-\alpha_l) = \sum_{m=0}^{5} \phi_m z^{5-m}\,,$$

where the $\phi$'s are standard symmetrical functions of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, and $\alpha_4$, then

$$\frac{\prod_{l=0}^4 (z-\alpha_l)}{(z-\alpha_k)} = \sum_{m=0}^{4} \psi_m z^{4-m}\,,$$

where

$$\psi_m = \alpha_k^m \phi_0 + \alpha_k^{m-1}\phi_1 + \cdots + \phi_m , \quad m = 0, 1, 2, 3, 4. \tag{6.43}$$

Taking the linear combinations of the rows in the determinant in (6.42), we can reduce the last two rows so that the $k^{\text{th}}$ entry in these rows becomes $\alpha_k^3 z_{1,2} + \alpha_k^4 - \alpha_k^3 \phi_1$, respectively. It is now easy to see that the latter determinant can be reduced to a Vandermonde determinant, and we obtain $M_{4,6} = \prod'_{k<l}(\alpha_k - \alpha_l) \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{(z_1-z_2)dz_1dz_2}{R_5(z_1)R_5(z_2)}$, where the prime denotes the absence of the considered $\alpha_j$ in a product or in a sum. The same arguments lead to

$$M_{4,j+1} = \prod_{k<l}'(\alpha_k - \alpha_l) \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{(z_1 - z_2)dz_1dz_2}{R_j(z_1)R_j(z_2)} . \tag{6.44}$$

Applying similar arguments, we find

$$M_{3,6} = \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{dz_1dz_2}{R_5(z_1)R_5(z_2)} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_4 \\ \alpha_0^3 & \alpha_1^2 & \cdots & \alpha_4^3 \\ \frac{\prod_{l=0}^4(z_1-\alpha_l)}{(z_1-\alpha_0)} & \frac{\prod_{l=0}^4(z_1-\alpha_l)}{(z_1-\alpha_1)} & \cdots & \frac{\prod_{l=0}^4(z_1-\alpha_l)}{(z_1-\alpha_4)} \\ \frac{\prod_{l=0}^4(z_2-\alpha_l)}{(z_2-\alpha_0)} & \frac{\prod_{l=0}^4(z_2-\alpha_l)}{(z_2-\alpha_1)} & \cdots & \frac{\prod_{l=0}^4(z_2-\alpha_l)}{(z_2-\alpha_4)} \end{pmatrix} , \tag{6.45}$$

so that the entries $k, m$, $k = 0, 1, 2, 3, 4$, $m = 5, 6$, of the last two rows of the determinant can be reduced to

$$\alpha_k^2(z_{m-4}^2 + \phi_1 z_{m-4} + \phi_2) + \alpha_k^4 . \tag{6.46}$$

Taking linear combinations of the last two rows, we obtain

$$M_{3,6} = -\prod_{k<l}'(\alpha_k - \alpha_l) \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{[(z_1^2 - z_2^2) + \phi_1(z_1 - z_2)]dz_1\, dz_2}{R_5(z_1)R_5(z_2)}$$

and, similarly,

$$M_{3,j+1} = -\prod_{k<l}'(\alpha_k - \alpha_l) \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{[(z_1^2 - z_2^2) + \phi_1(z_1 - z_2)]dz_1\, dz_2}{R_j(z_1)R_j(z_2)} . \tag{6.47}$$

Using the same routine, we obtain

$$M_{2,j+1} = \prod_{k<l}'(\alpha_k - \alpha_l) \int_{\hat{\gamma}_m} \int_{\hat{\gamma}_c} \frac{[(z_1^3 - z_2^3) + \phi_1(z_1^2 - z_2^2) + \phi_2(z_1 - z_2)]dz_1\, dz_2}{R_j(z_1)R_j(z_2)} . \tag{6.48}$$

Thus, according to (6.41),

$$|F_{jt}| = (-1)^{j+1} \prod_{k\neq j} \frac{h'(\alpha_k)}{2R(\alpha_k)} \times \prod_{k<l}{}' (\alpha_k - \alpha_l) \int_{\hat{\gamma}_c} \frac{[(z_1^2 - z_2^2) + (\alpha_j - \frac{1}{2}\sum \alpha_k)(z_1 - z_2)]dz_1\, dz_2}{R_j(z_1)R_j(z_2)}, \tag{6.49}$$

so that, using (6.2) and (6.39), one obtains

$$(a_j)_x = \frac{2R(\alpha_k)}{h'(\alpha_k)} \prod_{k\neq j} (\alpha_k - \alpha_j)^{-1} \left[ \left( \alpha_j - \frac{1}{2} \sum \alpha_k \right) \frac{D_{1,j}}{D} + \frac{D_{2,j}}{D} \right], \tag{6.50}$$

where

$$D = \det \begin{pmatrix} \int_{\hat{\gamma}_c} \frac{dz_2}{R(z_2)} & \int_{\hat{\gamma}_m} \frac{dz_1}{R(z_1)} \\ \int_{\hat{\gamma}_c} \frac{z_2 dz_2}{R(z_2)} & \int_{\hat{\gamma}_m} \frac{z_1 dz_1}{R(z_1)} \end{pmatrix}, \quad D_{k,j} = \det \begin{pmatrix} \int_{\hat{\gamma}_c} \frac{dz_2}{R_j(z_2)} & \int_{\hat{\gamma}_m} \frac{dz_1}{R_j(z_1)} \\ \int_{\hat{\gamma}_c} \frac{z_2^k dz_2}{R_j(z_2)} & \int_{\hat{\gamma}_m} \frac{z_1^k dz_1}{R_j(z_1)} \end{pmatrix}. \tag{6.51}$$

Let us now derive the equation for $\alpha_t$. Similarly to (6.41), we obtain

$$|F_{jt}| = (-1)^{j+4} \prod_{k\neq j} \frac{h'(\alpha_k)}{2R(\alpha_k)} \times \left( \frac{1}{2} \sum_{j=0}^{5} \alpha_j M_{3,j} - M_{2,j} - \left[ \frac{3}{8} \sum_{j=0}^{5} \alpha_j^2 + \frac{1}{4} \sum_{i<j} \alpha_i \alpha_j \right] M_{4,j} \right), \tag{6.52}$$

which, together with (6.44), (6.47), and (6.48), yields after some algebra

$$(a_j)_t = \frac{2R(\alpha_k)}{h'(\alpha_k)} \cdot \frac{(4\alpha_j^2 - 2\alpha_j \sum_k \alpha_k + \sum_{i<k} \alpha_i \alpha_k - \sum_k \alpha_k^2) D_{1,j} + 4(\alpha_j - \frac{1}{2}\sum \alpha_k) D_{2,j} + 4D_{3,j}}{D \prod_{k\neq j} (\alpha_k - \alpha_j)}. \tag{6.53}$$

## 6.4 Postbreak Solution in the Case $\mu \geq 2$

In this section we first show that in the case $\mu \geq 2$ solutions to (6.1) $\alpha_j(x,t)$, $j = 0, 1, \ldots, 5$, are bounded and stay away from the real axis for any compact set $S$ of the $(x,t)$, where $x \geq 0$, $t > 0$. The proof of the corresponding Theorem 6.8, given below, can be found in the forthcoming paper [34], since it fits naturally into its context. We then prove that $\alpha$ stays nondegenerate in the whole postbreak region.

Let us start by transforming the contour of integration $\hat{\gamma}$ in the four moment conditions $M_k$,

$$\frac{1}{\pi i} \int_{\hat{\gamma}} \frac{\zeta^k f'(\zeta) d\zeta}{R(\zeta)} = 0, \tag{6.54}$$

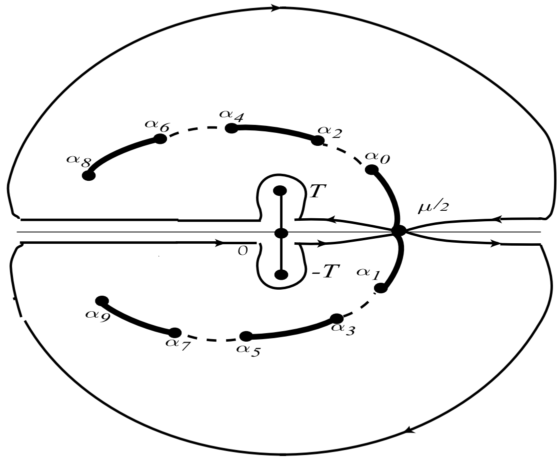


FIGURE 6.3. Deformation of $\hat{\gamma}$.

$k = 0, 1, 2, 3$, into the sum of two contour integrals encircling the upper half-plane and lower half-plane parts $\hat{\gamma}^{\pm}$ of $\hat{\gamma}$. The only point where these contours intersect $\mathbb{R}$ is $z = \frac{\mu}{2}$. The value of the integrals will not be changed if we add the constants $\pm\frac{i\pi}{2}$ to the integrand in $M_3$ in the upper and lower half-planes, respectively. Deform the contours $\hat{\gamma}^{\pm}$ to unions of $\mathbb{R}$ and the corresponding large radius semicircles $C^{\pm}$. In the case $\mu < 2$ we should also include the segments $[0, \pm T]$ to these deformations.

To compute the integrals in (6.54), we rewrite $f'(z)$ from (4.3) as

$$f'(z) = \frac{i\pi}{2} + \ln\frac{z}{z - \frac{\mu}{2}} + \frac{1}{2}\ln\left(1 - \frac{T^2}{z^2}\right) - x - 4tz = \frac{i\pi}{2} + \hat{f}(z)\,, \tag{6.55}$$

where the logarithm terms have cuts along $[0, \frac{\mu}{2}]$ and $[-T, T]$, respectively. Note that $\hat{f}(z)$ is analytic at infinity, so for this term the contour of integration could be deformed into a circle of some large radius, and the value could be computed through the residue at $z = \infty$; see Figure 6.3. Because of Schwarz reflection symmetry, the combined integrals over $\mathbb{R}$ yield

$$\begin{cases} \frac{1}{\pi}\int_{\mathbb{R}} \frac{\zeta^k \Im f'(\zeta)d\zeta}{R(\zeta)}, & k = 0, 1, 2, \\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{[\zeta^3 \Im f'(\zeta) - R(\zeta)]d\zeta}{R(\zeta)}, & k = 3. \end{cases} \tag{6.56}$$

Note that, according to (6.55), $\Im f'(\zeta) = \frac{\pi}{2}$ if $\zeta \le -T$ or $\zeta \ge \frac{\mu}{2}$, and $\Im f'(\zeta) = -\frac{\pi}{2}$ if $T \le \zeta < \frac{\mu}{2}$ and is equal to 0 on $(-T, T)$. Thus, replacing $R(z)$ by $|R(\zeta)|$ along $\mathbb{R}$ (i.e., replacing $R(\zeta)$ by $-R(z)$ for $\zeta > \frac{\mu}{2}$), we get $\Im f'(\zeta) = -\frac{\pi}{2}\operatorname{sign}\zeta$ if $|\zeta| \ge T$ and 0 otherwise. Therefore, for the solitonless case $\mu \ge 2$, the moment conditions (6.54) become

$$(M_0)\quad \int\limits_{|\zeta|\geq T} \frac{\operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 0\,, \qquad (M_1)\quad \int\limits_{|\zeta|\geq T} \frac{\zeta \operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 8t\,,$$

$$(M_2)\quad \int\limits_{|\zeta|\geq T} \frac{\zeta^2 \operatorname{sign}\zeta\, d\zeta}{|R(\zeta)|} = 2x + 8t\sum_{j=0}^{2} a_{2j}\,, \tag{6.57}$$

$$(M_3)\quad \int\limits_{|\zeta|\geq T} \frac{[\zeta^3 \operatorname{sign}\zeta - |R(\zeta)|]d\zeta}{|R(\zeta)|} = 2x\sum_{j=0}^{2} a_{2j} + 8tK(\alpha) - \mu + 2T\,,$$

where $\alpha_j = a_j + ib_j$, the quadratic form $K(\alpha) = \frac{1}{2}\sum_{j<k}(a_{2j}+a_{2k})^2 - \frac{1}{2}\sum_{j=0}^{2} b_{2j}^2$, and

$$|R(z)| = \prod_{j=0}^{2} |z - \alpha_{2j}|\,, \quad z \in \mathbb{R}\,. \tag{6.58}$$

One has to take the Cauchy principal value of the integrals in $M_2$ and $M_3$. In the case $\mu < 2$ the value of $T$ becomes purely imaginary. Thus, we have to integrate the integrals on the left-hand sides of $M_k$ along all of $\mathbb{R}$ and add the integrals $2i\int_0^T \Im\frac{\zeta^k}{R(\zeta)}\,d\zeta$ to the corresponding right-hand sides.

PROOF OF LEMMA 5.2: In our present notation, we have $z_0 = \alpha_2 = \alpha_4$ and $\alpha = \alpha_0$. The condition $M_0$ immediately implies $\Re\alpha_2 \leq 0$. It is also clear that $\Re\alpha_2 = 0$ requires $\Re\alpha_0 = 0$, which, according to results of Sections 4.5 and 4.6, takes place only at the breaking point $x = 0$, $t = \frac{1}{2(\mu+2)}$. But if $x > 0$, then $\Re\alpha_0 > 0$ implies $\Re\alpha_2 < 0$. The last statement of Lemma 5.2 also follows from $M_0$ immediately. This completes the proof. □

THEOREM 6.8 *If $\mu \geq 2$, then for any compact set $S$ of the parameters $x \geq 0$, $t > 0$, the values of $\alpha_j$ satisfying* (6.57) *are bounded and separated from* $\mathbb{R}$.

Theorem 6.8 shows that in the case $\mu \geq 2$ a collision of $\alpha$'s cannot happen anywhere on the real axis, including the point $z = \infty$. Note that the assumption $\mu \geq 2$ rules out any singularities of $f(z)$ in the upper half-plane. Let us show that $\alpha$ stays nondegenerate in the whole postbreak region. Then, according to the evolution theorem (Theorem 3.2), conditions (2.19) with $N = 1$ can be extended from some region above the breaking curve (see Theorem 6.6) to the whole postbreak region.

THEOREM 6.9 *In the case $\mu \geq 2$ the solutions of equations* (6.1) *satisfying the boundary condition $\alpha_2 = \alpha_4$ and $\Re\alpha_0 \geq 0$ on the breaking curve have a unique continuation into the entire region above the breaking curve $t = t_0(x)$. Moreover, all the conditions* (2.19) *with $N = 1$ are satisfied in that region.*

PROOF: According to Theorem 6.4 and Lemma 6.7, such a solution exists in a region $Q^+$ above the breaking curve, where the $\alpha$'s are bounded and distinct. If the $\alpha$'s stay distinct, conditions (2.19) with $N = 1$ can break only if one of the inequalities in (2.19) breaks. That would require another branch of a zero level curve of $\Im h$ to intersect the contour $\gamma$. But in the case $\mu \geq 2$ such an extra branch has no place to originate from, since three branches of the zero level curve of $\Im h$ going to infinity already exist. (Note that in the case $\mu < 2$ such an extra branch could originate from the vertical cut $[-T, T]$.) Since the $\alpha$'s stay bounded and away from the real axis for all $t \geq 0$, it remains only to prove that after the break the points $\alpha_0$, $\alpha_2$, and $\alpha_4$ stay distinct.

Let us first use (6.57) to show that $\alpha_0$, $\alpha_2$, and $\alpha_4$ cannot become equal after the break. Indeed, if they are equal, then, according to $M_0$, the common point $\alpha_0 = ib$ is on the positive imaginary semiaxis. Then, according to $M_2$, we obtain $x = 0$. Direct computations show that the remaining conditions $M_1$ and $M_3$ determine the corresponding $b$ and $t$ uniquely. But such values $b_0 = \sqrt{\mu + 2}$ and $t_0 = \frac{1}{2(\mu+2)}$ have already been determined for the first break. Thus, the case $\alpha_0 = \alpha_2 = \alpha_4$ is excluded.

Let $\alpha_4$ denote the last branch point on the arc of the zero level curve of $\Im h = 0$ in the upper half-plane connecting $\frac{\mu}{2}$ with $-\frac{\mu}{2}$. Then the only possibilities left for collision are between $\alpha_2$ and $\alpha_4$ and between $\alpha_0$ and $\alpha_2$. Indeed, the collision between $\alpha_0$ and $\alpha_4$ leads to a contradiction since it would imply the existence of a nontrivial (since the angles at $\alpha_2$ are $\frac{\pi}{3}$) loop $\alpha_0$–$\alpha_2$–$\alpha_4$ of the zero level curve of the harmonic function $\Im h$.

Suppose that some point $(x_1, t_1)$ in the postbreak region is a point of collision of $\alpha_2$ with $\alpha_0$ or $\alpha_4$ (double point), denoted by $z_0$, where $\tilde{\alpha}$ denotes the remaining branch point. Then the contour $\gamma^+$ consists of the main arc $\gamma_m$ connecting $\frac{\mu}{2}$ with $\tilde{\alpha}$ and the complementary arc $\gamma_c$ connecting $\tilde{\alpha}$ with $-\frac{\mu}{2}$. The double point $z_0$ is also on the contour $\gamma^+$. Note that, according to $M_0$, $\tilde{\alpha}$ and $z_0$ are situated in opposite half-planes. In the case of the $\alpha_2$–$\alpha_4$ collision, $\tilde{\alpha} = \alpha_0$ and $z_0 = \alpha_2 = \alpha_4$ lies on $\gamma_c$. In the case of the $\alpha_0$–$\alpha_2$ collision, $\tilde{\alpha} = \alpha_4$ and $z_0 = \alpha_0 = \alpha_2$ lie on $\gamma_m$.

Let $t_0(\hat{x}) = t_1$ where $\hat{x} > x_1$, and now let $(x_1, t_1)$ be the first collision point on the line $t = t_1$ to the left of the breaking curve; see Figure 6.4.

According to the degeneracy theorem (Theorem 3.1), the point $\tilde{\alpha}$ satisfies equations (4.1), so that either $\Re\tilde{\alpha} > 0$ or $\Re\tilde{\alpha} < 0$; see Theorem 4.1. Let us show that the former case is not possible. Indeed, let $\tilde{\alpha}(\hat{x}, t_1)$ and $z_0(\hat{x}, t_1)$ be the branch point and the double point at the point $(\hat{x}, t_1)$, respectively. Let us consider a curve $\theta$ passing from some fixed point on the interval $(-\frac{\mu}{2}, \frac{\mu}{2})$ to $i\infty$ through $z_0$ such that $\Im h_0(z) < 0$ for all $z \in \theta$ except $z = z_0$; see Figure 6.5. According to Lemma 4.5, $\Im h_0(z) < 0$ for all $z \in \theta$, $\Re z < 0$, when $x < \hat{x}$ and $t = t_1$. So, there can be no double point $z_0(x, t_1)$ in the left half-plane. But, according to $M_0$, $\Re z_0(x, t_1) < 0$. The obtained contradiction shows that the case $\Re\tilde{\alpha} > 0$ is excluded. (In the case when $\gamma_m$ "veers" into the left half-plane, $z_0(\hat{x}, t_1)$ cannot be inside a "loop" of $\gamma_m$ where

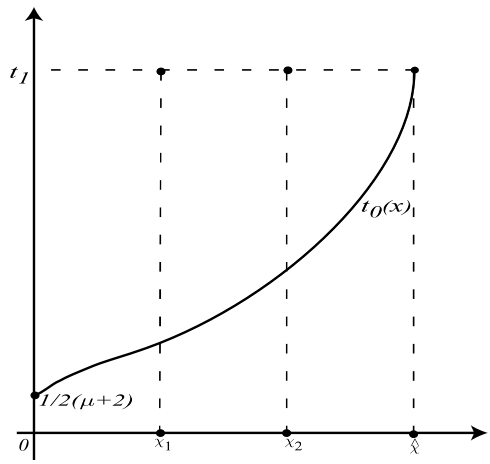


FIGURE 6.4. Points $x_1$ and $\bar{x}$.

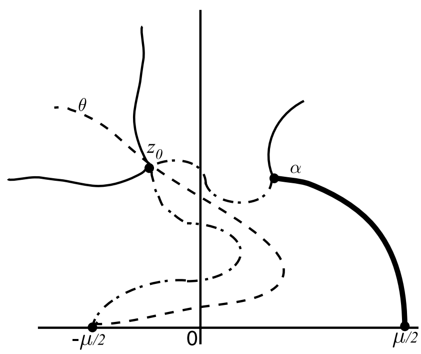


FIGURE 6.5. Curve $\theta$.

$\Im h_x(z_0) < 0$; see Figure 4.4. Indeed, it would contradict the fact that $\Im h(z) > 0$ for all $z = a + i\eta$, $\eta > b$; see Figure 4.7 and the proof of Theorem 4.10.)

Thus, we have to consider the case $\Re\tilde{\alpha} < 0$ and $\Re z_0 > 0$. If the double point $z_0 \in \gamma_m$ (see Figure 6.6), then the conditions (2.19) with $N = 0$ are satisfied for all $x \in [0, x_1)$, $t = t_1$, due to the fact that $\Im h_x(z) > 0$ above $\gamma$ when $\Re z > 0$ and due to the topology of zero level curves of $\Im h_0(z)$.

If the double point $z_0 \in \gamma_c$ (see Figure 6.7), then $\Im h_x(z_0) < 0$, since otherwise the conditions (2.19) are satisfied with both $N = 2$ and $N = 0$ when $x \in (x_1, \hat{x})$, $t = t_1$. Then, according to Lemma 4.5, the point $z_0$ lies "below" $\gamma_m$. So, conditions (2.19) with $N = 0$ are satisfied for some $x < x_1$ that are close to $x_1$. We can now claim that these conditions are valid for all $x \in [0, x_1)$, $t = t_1$. Indeed, the only way for them to have (the first) break at some point $(x, t_1)$, where $x \in [0, x_1)$, is for the branch of the zero level curve of $\Im h_0$ that starts and ends at infinity to reconnect itself to $\gamma_c^+$ at some point $z_0(x, t_1)$. Note that reconnection to $\gamma_m^+$ is impossible due

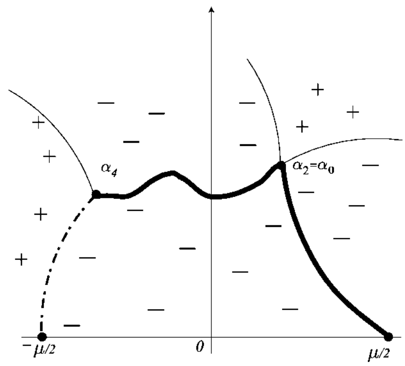


FIGURE 6.6. Case $\Re\hat{a} < 0$, $z_0 \in \gamma_m$.

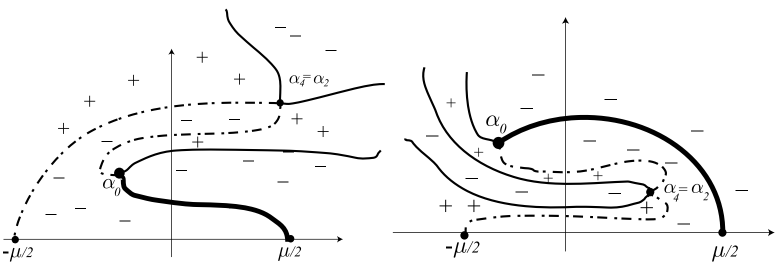


FIGURE 6.7. Cases $\Re\hat{a} < 0$, $z_0 \in \gamma_c$.

to the sign structure of $\Im h_0$. Since $\Re z_0(x, t_1) > 0$, we have that $\Im h_x$ at $(z_0(x, t_1))$ is negative. Thus, reconnection at $z_0(x, t_1)$ is not possible, since it would mean that conditions (2.19) with $N = 0$ were violated before the point $(x, t_1)$ (for some larger $x$).

Thus, any collision of $\alpha$'s at the point $(x_1, t_1)$ above the breaking curve implies that conditions (2.19) with $N = 0$ are satisfied for $x = 0$ and some $t > t_0(0)$, where $t_0(0) = \frac{1}{2(\mu+2)}$. Let us show that this is impossible.

At the breaking point $(0, t_0)$ the branch point $\alpha = i\sqrt{\mu + 2}$. The topology of zero level curves of $\Im h$ is shown on Figure 6.8.

After the breaking time, $\alpha$ moves either to the right or to the left half-plane according to Theorem 4.1. In the former case, there can be no zero level curve of $\Im h_0(z)$ connecting $\alpha$ and $-\frac{\mu}{2}$ for $t > t_0$ due to the "sea of minuses," since $\Im h_0(i\sqrt{\mu + 2}) < 0$. In the latter case, there can also be no zero level curve of $\Im h_0(z)$ connecting $\alpha$ and $\frac{\mu}{2}$ for $t > t_0$. Indeed, this level curve cannot pass through the positive sector above $i\sqrt{\mu + 2}$, because $\Im h_t(z)$ is positive there, and cannot pass through the negative sector below $i\sqrt{\mu + 2}$, because $\Im h_t(z)$ is negative there. At the point $i\sqrt{\mu + 2}$ itself, the function $\Im h_0$ attains either positive or negative values, depending on what sheet of the Riemann surface of $h_0$ we are considering.

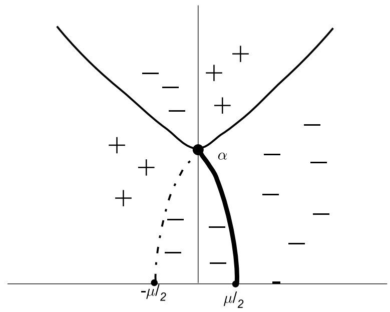


FIGURE 6.8. Level curves of $\Im h$ at $x = 0,\ t = t_0(0)$.

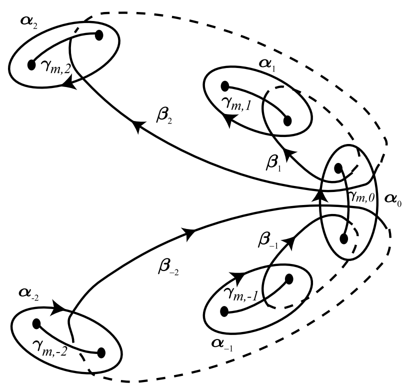


FIGURE 7.1. Basic cycles.

Thus, conditions (2.19) with $N = 0$ cannot be satisfied for $x = 0$ and any $t > t_0(0)$. The proof is complete. □

## 7 Solution of the Model RHP

### 7.1 Riemann-Hilbert Formalism for the Inverse Scattering Problem

In this section we derive the explicit solution of the *model* RHP $P^{(\text{mod})}$. With the cuts along the main arcs $\gamma_m$, we construct a hyperelliptic surface with a canonical homology basis as shown in Figure 7.1.

The dotted curves pass through the second sheet. The cycles $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j\}_{|j|=1}^{N}$ form a canonical homology basis, where $2N$ is the genus of the surface, $N > 0$. Now we are aiming at eliminating the diagonal part of the jump matrix $V^{(\text{mod})}$ by introducing a function $\tilde{g}(z)$ as follows: Set $\tilde{m}^{(\text{mod})} = m^{(\text{mod})} e^{-2(i/\varepsilon)\tilde{g}\sigma_3}$. Then the RHP $P^{(\text{mod})}$ is transformed into the RHP $\widetilde{P}^{(\text{mod})}$: $\tilde{m}_+^{(\text{mod})} = \tilde{m}_-^{(\text{mod})} \tilde{V}^{(\text{mod})}$, where

$$\tilde{V}^{(\text{mod})} = e^{2\frac{i}{\varepsilon}\tilde{g}_-\sigma_3} V^{(\text{mod})} e^{-2\frac{i}{\varepsilon}\tilde{g}_+\sigma_3}\,. \tag{7.1}$$

On the main arcs the matrix $\tilde{V}^{(\mathrm{mod})}$ is

$$(7.2)\qquad \tilde{V}^{(\mathrm{mod})} = \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W-\tilde{g}_- - \tilde{g}_+)} \\ -e^{\frac{2i}{\varepsilon}(W-\tilde{g}_- - \tilde{g}_+)} & 0 \end{pmatrix},$$

whereas on the complementary arcs $\tilde{V}^{(\mathrm{mod})}$ is

$$(7.3)\qquad \tilde{V}^{(\mathrm{mod})} = e^{2\frac{i}{\varepsilon}(\Omega+\tilde{g}_- - \tilde{g}_+)\sigma_3}\,.$$

It is now clear that to eliminate the diagonal part we have to set the RHP for $\tilde{g}$ as:

$$(7.4)\qquad \begin{cases} \tilde{g}_+(z) - \tilde{g}_-(z) = \Omega & \text{when } z \in \gamma_c \\ \tilde{g}_+(z) + \tilde{g}_-(z) = \Delta & \text{when } z \in \gamma_m\,. \end{cases}$$

Here $\Omega = (\Omega_{-N}, \dots, \Omega_{-1}, \Omega_1, \dots, \Omega_N)^{\mathsf{T}}$, where the real numbers $\Omega_k$, $k > 0$, are defined in (3.8) and, according to the symmetry condition, $\Omega_{-k} = \Omega_k$. The $(2N+1)$–dimensional vector $\Delta$ consists of $2N$ constants $\Delta_k \in \mathbb{C}$, $k = \pm 1, \pm 2, \dots, \pm N$, to be determined and $\Delta_0 = 0$. In fact, these constants are determined uniquely by the solvability of the scalar RHP (7.4). Here and henceforth we keep the notation $\Delta$ for the vector $\Delta = (\Delta_{-N}, \dots, \Delta_{-1}, \Delta_1, \dots, \Delta_N)^{\mathsf{T}}$.

Now the RHP $\widetilde{P}^{(\mathrm{mod})}$ has no jumps on the complementary arcs, whereas on the main arcs $\gamma_{m,k}$, $k = \pm 1, \pm 2, \dots, \pm N$,

$$(7.5)\qquad \tilde{V}^{(\mathrm{mod})} = \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_k-\Delta_k)} \\ -e^{\frac{2i}{\varepsilon}(W_k-\Delta_k)} & 0 \end{pmatrix},$$

and $\tilde{V}^{(\mathrm{mod})} = i\sigma_2$ on $\gamma_0 = \gamma_{m,0}^+ \cup \gamma_{m,0}^-$. Here $\gamma_{m,k} = \gamma_{m,k}^+$ for $k > 0$ and $\gamma_{m,k} = \gamma_{m,-k}^-$ for $k < 0$. The same convention is applied to $\gamma_{c,k}$.

Next we proceed to determine $\Delta$. Using the convention of (2.19), i.e., that $\Delta = \Delta_k$ and $\Omega = \Omega_k$ when $z \in \gamma_{m,k}$ or $z \in \gamma_{c,k}$, respectively, $k = \pm 1, \pm 2, \dots, \pm N$, the function $\tilde{g}(z)$ has the Cauchy representation

$$(7.6)\qquad \tilde{g}(z) = \frac{R(z)}{2\pi i}\left[\int_{\gamma_m} \frac{\Delta}{(\zeta - z)R_+(\zeta)}\,d\zeta + \int_{\gamma_c} \frac{\Omega}{(\zeta - z)R(\zeta)}\,d\zeta\right].$$

By requiring that $\tilde{g}(z)$ in (7.6) be analytic at $\infty$, we obtain the system of moment conditions

$$(7.7)\qquad \int_{\gamma_m} \frac{\Delta}{R_+(\zeta)}\zeta^j d\zeta + \int_{\gamma_c} \frac{\Omega}{R(\zeta)}\zeta^j d\zeta = 0\,, \quad j = 0, 1, \dots, 2N-1.$$

If this system is uniquely solvable, then the symmetries of $\Omega$ and of $\zeta^j d\zeta / R(\zeta)$ would produce the symmetry $\Delta_k = \bar{\Delta}_{-k}$. To solve system (7.7), we may use holomorphic differentials. The Riemann surface theory gives a basis $\omega = (\omega_{-N},$

$\omega_{-N+1},\dots,\omega_{-1},\omega_1,\dots,\omega_N)$ of holomorphic differentials dual to the $\alpha$ cycles; see Figure 7.1.

$$\int_{\boldsymbol{\alpha}_j} \omega_k = \delta_{jk}\,, \quad |j|,|k| = 1,\dots,N, \tag{7.8}$$

where $\delta_{jk}$ denotes the Kronecker symbol. Each of these differentials $\omega_k$ has the form $\omega_k = \frac{P_k(z)}{R(z)}\,dz$, where $P_k(z)$ is a polynomial of degree less than $2N$. Thus the linear combinations of (7.7) produce the system

$$\int_{\gamma_m} \Delta\omega_j(\zeta) + \int_{\gamma_c} \Omega\omega_j(\zeta) = 0\,, \quad |j| = 1,\dots,N. \tag{7.9}$$

Using (7.8), we obtain solutions to (7.9) as

$$\Delta_j = -2\int_{\gamma_c} \Omega\omega_j\,, \quad |j| = 1,\dots,N. \tag{7.10}$$

With the choice of $\Delta$ given by (7.10), the RHP $\widetilde{P}^{(\mathrm{mod})}$ is supported only on $\gamma_m$. Define $\widetilde{W} \in \mathbb{C}^{2N}$ as $\widetilde{W} = W - \Delta$. Then $\widetilde{W}$ has the symmetry

$$\overline{\widetilde{W}}_k = \widetilde{W}_{-k}\,. \tag{7.11}$$

The jump matrix $\tilde{V}^{(\mathrm{mod})}$ is now given by

$$\tilde{V}^{(\mathrm{mod})} = \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}\widetilde{W}} \\ -e^{\frac{2i}{\varepsilon}\widetilde{W}} & 0 \end{pmatrix} \tag{7.12}$$

on $\gamma_m$. This RHP is very similar to that from [14] except here we have in general nonreal $\widetilde{W}$. However, we will adopt a slightly different approach for this RHP.

The Riemann period matrix is defined as

$$\tau = (\tau_{kj}) = \left(\int_{\boldsymbol{\beta}_j} \omega_k\right), \quad |j|,|k| = 1,\dots,N. \tag{7.13}$$

$\tau$ is known to be symmetric and pure imaginary, and $-i\tau$ is positive definite. The Riemann theta function is defined as

$$\theta(s) = \sum_{l\in\mathbb{Z}^{2N}} e^{2\pi i(l,s)+\pi i(l,\tau l)}\,, \quad s \in \mathbb{C}^{2N}\,. \tag{7.14}$$

The theta function satisfies

$$\theta(s) = \theta(-s)\,, \tag{7.15a}$$

$$\theta(s+e_j) = \theta(s)\,, \tag{7.15b}$$

$$\theta(s+\tau_j) = e^{\pm 2\pi i s_j - \pi i \tau_{jj}}\theta(s)\,, \tag{7.15c}$$

where $e_j$ is the $j^{\text{th}}$ column of $I_{2N\times 2N}$ and $\tau_j = \tau e_j$. We call $Ł = \mathbb{Z}^{2N} + \tau\mathbb{Z}^{2N}$ the period lattice of the theta function.

Denote $u(z) = \int_{\alpha_1}^{z} \omega$ and

$$(7.16)\quad \mathcal{M}(z,d) \equiv (\mathcal{M}_1, \mathcal{M}_2) = \left( \frac{\theta(u(z) - \frac{\widehat{W}}{2\pi} + d)}{\theta(u(z)+d)}, \frac{\theta(-u(z) - \frac{\widehat{W}}{2\pi} + d)}{\theta(-u(z)+d)} \right),$$

where $\widehat{W} = -2\frac{i}{\varepsilon}(\widetilde{W}_{-N}, \dots, \widetilde{W}_{-1}, \tilde{W}_1, \dots, \widetilde{W}_N)^{\mathsf{T}}$ and $d \in \mathbb{Z}^{2N}$ is a vector to be determined. Although $u(z)$ is multivalued, $\mathcal{M}(z,d)$ is single valued and meromorphic away from $\gamma_m$. Moreover, $\mathcal{M}$ satisfies

$$(7.17)\quad \mathcal{M}_+ = \mathcal{M}_- \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}\widetilde{W}} \\ e^{\frac{2i}{\varepsilon}\widetilde{W}} & 0 \end{pmatrix}$$

on $\gamma_m$ (see [41]).

Introduce

$$\lambda(z) = \left( \frac{z-\alpha_0}{z-\alpha_1} \prod_{j=1}^{N} \frac{(z-\alpha_{4j})(z-\alpha_{4j-1})}{(z-\alpha_{4j-2})(z-\alpha_{4j+1})} \right)^{\frac{1}{4}}$$

with branch cuts along $\gamma_m$. We choose the branch of $\lambda(z)$ such that $\lim_{z\to\infty} \lambda(z) = 1$ and that $\lambda_+ = i\lambda_-$ on $\gamma_m$. One can directly verify that

$$(7.18)\quad \mathcal{N}_+ = \mathcal{N}_- \tilde{V}^{(\mathrm{mod})},$$

where

$$(7.19)\quad \mathcal{N}(z) = \lambda^{\sigma_3}(z) \begin{pmatrix} \mathcal{M}_1(z,d_1) & -i\mathcal{M}_2(z,d_1) \\ -i\mathcal{M}_1(z,d_2) & \mathcal{M}_2(z,d_2) \end{pmatrix}.$$

We need to choose vectors $d_1$ and $d_2$ such that $\mathcal{N}(z)$ is locally $L^2$ at the branch points $\alpha_k$, $k = 0, 1, \dots, 4N+1$. According to the theory of Riemann surfaces [17], we may set

$$(7.20)\quad d_1 = \int_{\gamma_m \setminus \gamma_{m,0}} \omega = \frac{1}{2}(1, 1, \dots, 1)^{\mathsf{T}} \in \mathbb{C}^{2N}$$

so that $\theta(u(z) \pm d_1)$ has exactly $2N$ zeros (of square root type) at the beginnings of arcs $\gamma_{m,j}$, $|j| = 1, \dots, N$. Note that $d_1 = -\int_{\gamma_{m,0}} \omega$ because $\omega$ is analytic at infinity. Similarly, we may set $d_2 = 0$ so that $\theta(u(z) \pm d_2)$ has exactly $2N$ zeros (of square root type) at the ends of arcs $\gamma_{m,j}$, $|j| = 1, \dots, N$. According to (7.19), $\mathcal{N}(z)$ has at worst quarter root type singularities at the endpoints $\alpha_k$ of the main arcs. Thus, we conclude that $\mathcal{N}(z)$ is a fundamental solution to the RHP $\tilde{V}^{(\mathrm{mod})}$, which has not yet been normalized at infinity:

$$\mathcal{N}(\infty) = \begin{pmatrix} \mathcal{M}_1(\infty, d_1) & -i\mathcal{M}_2(\infty, d_1) \\ -i\mathcal{M}_1(\infty, 0) & \mathcal{M}_2(\infty, 0) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\theta(u(\infty) - \frac{\widehat{W}}{2\pi} + d_1)}{\theta(u(\infty) + d_1)} & -i\dfrac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d_1)}{\theta(u(\infty) + d_1)} \\ -i\dfrac{\theta(u(\infty) - \frac{\widehat{W}}{2\pi})}{\theta(u(\infty))} & \dfrac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi})}{\theta(u(\infty))} \end{pmatrix}. \tag{7.21}$$

Note that some symmetry in (7.21) can be observed by writing $u(\infty)+d_1 = \int_{\alpha_0}^{\infty} \omega$. In order to find the final, i.e., the normalized, solution to the RHP, we need to know $\tilde{g}(\infty)$.

As shown in [17], there exists a unique monic polynomial $P(z)$ of degree $2N$ such that

$$\int_{\boldsymbol{\alpha}_j} \frac{P(z)}{R(z)}\, dz = 0\,, \quad |j| = 1, \dots, N. \tag{7.22}$$

This equation, combined with (7.6) and (7.7), yields

$$\begin{aligned} \tilde{g}(\infty) &= \frac{1}{2\pi i}\int_{\gamma_m} \frac{\Delta}{R_+(\zeta)}\zeta^{2N}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma_c} \frac{\Omega}{R(\zeta)}\zeta^{2N}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_m} \frac{\Delta}{R_+(\zeta)} P(\zeta) d\zeta + \frac{1}{2\pi i}\int_{\gamma_c} \frac{\Omega}{R(\zeta)} P(z) d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_c} \frac{\Omega P(\zeta)}{R(\zeta)}\, d\zeta\,. \end{aligned} \tag{7.23}$$

According to (2.33), $m^{(\mathrm{mod})}(\infty) = e^{2\frac{i}{\varepsilon}g(\infty)\sigma_3}$ and $\tilde{m}^{(\mathrm{mod})} = m^{(\mathrm{mod})}e^{-2\frac{i}{\varepsilon}\tilde{g}\sigma_3}$; hence $\tilde{m}^{(\mathrm{mod})}(z) = e^{2\frac{i}{\varepsilon}[g(\infty)-\tilde{g}(\infty)]\sigma_3}\mathcal{N}^{-1}(\infty)\mathcal{N}(z)$. Then in the limit $\varepsilon \to 0$

$$m^{(2)}(z) = e^{2\frac{i}{\varepsilon}[g(\infty)-\tilde{g}(\infty)]\sigma_3}\mathcal{N}^{-1}(\infty)\mathcal{N}(z)e^{-2\frac{i}{\varepsilon}[g(z)-\tilde{g}(z)]\sigma_3}\,. \tag{7.24}$$

From (7.21)

$$\det\mathcal{N}(\infty) = \frac{\theta(u(\infty) - \frac{\widehat{W}}{2\pi} + d_1)\theta(u(\infty) + \frac{\widehat{W}}{2\pi}) + \theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d_1)\theta(u(\infty) - \frac{\widehat{W}}{2\pi})}{\theta(u(\infty) + d_1)\theta(u(\infty))}\,. \tag{7.25}$$

Alternatively, taking into account that $\det\mathcal{N}(z)$ is constant in $z$, we obtain

$$\begin{aligned} \det\mathcal{N}(\infty) = \det\mathcal{N}(\alpha_1) &= \frac{\theta(-\frac{\widehat{W}}{2\pi} + d_1)\theta(\frac{\widehat{W}}{2\pi}) + \theta(\frac{\widehat{W}}{2\pi} + d_1)\theta(-\frac{\widehat{W}}{2\pi})}{\theta(d_1)\theta(0)} \\ &= 2\frac{\theta(-\frac{\widehat{W}}{2\pi} + d_1)\theta(\frac{\widehat{W}}{2\pi})}{\theta(d_1)\theta(0)}\,. \end{aligned} \tag{7.26}$$

Here we have used the fact that $d_1$ is a half period.

We now compute

$$(7.27)\quad (\mathcal{N}^{-1}(\infty)\mathcal{N}(z))_{12} = i\frac{\lambda^{-1}\mathcal{M}_2(\infty, d_1)\mathcal{M}_2(z, 0) - \lambda\mathcal{M}_2(\infty, 0)\mathcal{M}_2(z, d_1)}{\det\mathcal{N}(\infty)} \equiv \frac{A}{z} + O(z^{-2})$$

as $z \to \infty$. Direct calculations show that for any period $2d \in \mathbb{C}^{2N}$

$$\lambda(z) = 1 + \frac{i}{2z}\Im\sum_{j=1}^{N}(\alpha_{4j-2} - \alpha_{4j}) + O(z^{-2}),$$

$$(7.28)\quad \begin{aligned} \mathcal{M}_2(z, d) &= \mathcal{M}_2(\infty, d)\left[1 - \frac{1}{z}\nabla\ln\frac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d)}{\theta(u(\infty) + d)} \cdot \omega^0 + O(z^{-2})\right] \\ &= \mathcal{M}_2(\infty, d)\left[1 - \frac{1}{z}\left(\frac{\nabla\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d)}{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d)} - \frac{\nabla\theta(u(\infty) + d)}{\theta(u(\infty) + d)}\right) \cdot \omega^0 + O(z^{-2})\right] \end{aligned}$$

as $z \to \infty$, where $\omega^0$ is the leading coefficient for $\frac{R\omega}{dz}$, i.e., $\omega \sim \frac{\omega^0}{z^2}\,dz$. Taking into account (7.26)–(7.28) and (7.15), we obtain

$$(7.29)\quad \begin{aligned} A &= \frac{\mathcal{M}_2(\infty, d_1)\mathcal{M}_2(\infty, 0)}{\det\mathcal{N}(\infty)} \times \left[\Im\sum_{j=1}^{N}(\alpha_{4j-2} - \alpha_{4j}) + i\nabla\ln\frac{\mathcal{M}_2(u(\infty), d_1)}{\mathcal{M}_2(u(\infty), 0)} \cdot \omega^0\right] \\ &= \frac{\theta(0)\theta(d_1)}{2\theta(u(\infty))\theta(u(\infty) + d_1)} \times \frac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d_1)\theta(u(\infty) + \frac{\widehat{W}}{2\pi})}{\theta(\frac{\widehat{W}}{2\pi} + d_1)\theta(\frac{\widehat{W}}{2\pi})} \\ &\quad \times \left[\Im\sum_{j=1}^{N}(\alpha_{4j-2} - \alpha_{4j}) + i\nabla\ln\frac{\theta(u(\infty) + \frac{\widehat{W}}{2\pi} + d_1)\theta(u(\infty))}{\theta(u(\infty) + d_1)\theta(u(\infty) + \frac{\widehat{W}}{2\pi})} \cdot \omega^0\right]. \end{aligned}$$

If we write $m^{(2)} = I + m_1^{(2)}/z + O(z^{-2})$ as $z \to \infty$, then the leading asymptotic term for the solution is given by

$$(7.30)\quad q_0(x, t, \varepsilon) = -2\big(m_1^{(2)}\big)_{12} = -2Ae^{\frac{4i}{\varepsilon}[g(\infty) - \tilde{g}(\infty)]},$$

where expressions for $A$ and $\tilde{g}(\infty)$ are given in (7.23) and (7.29), respectively, and according to (3.17),

$$g(\infty) = \frac{1}{2\pi i}\int_{\gamma_m} \frac{f(\zeta)+W}{R_+(\zeta)} P(\zeta)d\zeta + \frac{1}{2\pi i}\int_{\gamma_c} \frac{\Omega}{R(\zeta)} P(\zeta)d\zeta \,. \tag{7.31}$$

Let us now summarize the results of the present section. Let the values $(x,t)$ for the initial value problem (1.1)–(1.2) belong to a genus $2N$ region. Let $\alpha_j(x,t)$, $j = 0, 1, \dots, 4N+1$, be the branch points that determine the corresponding Riemann surface $\mathcal{R}$, and let the corresponding vectors $W, \Omega \in \mathbb{R}^{2N}$ be determined according to (3.8), (7.10), and the comments after (7.4).

THEOREM 7.1 *In the region of genus* $2N$ *the leading-order term of the solution* (*as* $\varepsilon \to 0$) *to* (1.1)–(1.2) *has the form of*

$$q_0(x,t,\varepsilon) = -2Ae^{\frac{2}{\varepsilon\pi}\int_{\gamma_m} \frac{f(\zeta)P(\zeta)}{R(\zeta)}d\zeta} \,. \tag{7.32}$$

*Here the order* $2N$ *polynomial* $P(z)$ *is determined by* (7.22)*, and* $A$ *is given by* (7.29)*, where*

- *the vector* $\omega^0$ *is a leading coefficient of the basic holomorphic differentials* $\omega$ *for* $\mathcal{R}$*, dual to* $\boldsymbol{\alpha}$*,*
- $u(z) = \int_{\alpha_1}^{z} \omega$*, and*
- $\widehat{W} = -2\frac{i}{\varepsilon}(W + 2\int_{\gamma_c} \Omega\omega)$ *and* $d_1 = \frac{1}{2}(1,1,\dots,1)^{\mathsf{T}}$.

## 8 Alternative Formulae for Solving the Model RHP

An alternative way of solving the model PHP $P^{(\mathrm{mod})}$ (2.31) is based on the observation that, due to the Schwarz symmetry of the problem, the constants $W_j$ and $\Omega_j$ attain the same values on the arcs $\gamma_{m,j}^{\pm}$ and $\gamma_{c,j}^{\pm}$, respectively. Therefore, the contour $\Sigma^{(\mathrm{mod})}$ can be deformed, as shown in Figure 8.1, into the new contour $\tilde{\Sigma}^{(\mathrm{mod})}$ that consists of $2N+1$ vertical segments $\tilde{\nu}_k$ connecting the branch points $\alpha_{2k}$ and $\bar{\alpha}_{2k} = \alpha_{2k+1}$, $k = 0, 1, \dots, 2N$. The thin solid and dotted lines in Figure 8.1 show deformations of main and complementary arcs of $\Sigma^{(\mathrm{mod})}$, respectively. (In general, the order of $\alpha_{2k}$ along the contour $\Sigma^{(\mathrm{mod})+}$ does not necessarily coincide with the order of $-\Re\alpha_{2k}$, $k = 0, 1, \dots, 2N$, as shown in Figure 8.1. However, if necessary, we can always deform some vertical cuts $\tilde{\nu}_k$ into curves, connecting the same endpoints and symmetric with respect to the real axis, so that the order in which they intersect the real axis coincides with the order of $\alpha_{2k}$ along the contour $\Sigma^{(\mathrm{mod})+}$.)

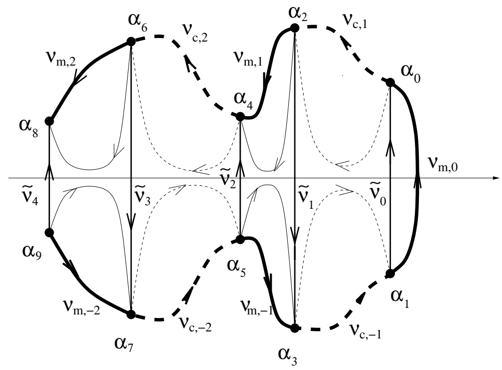


FIGURE 8.1. Deformation of the contour $\Sigma^{(\mathrm{mod})}$ into $\tilde{\Sigma}^{(\mathrm{mod})}$.

We can set an equivalent RHP for the matrix function $m^{(\mathrm{mod})}(z)$ with the contour $\tilde{\Sigma}^{(\mathrm{mod})}$ and constant (in $z$) jump matrices

$$\begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}W_j} \\ -e^{\frac{2i}{\varepsilon}W_j} & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{2i}{\varepsilon}\Omega_{j+1}} & 0 \\ 0 & e^{-\frac{2i}{\varepsilon}\Omega_{j+1}} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_j-\Omega_{j+1})} \\ -e^{\frac{2i}{\varepsilon}(W_j-\Omega_{j+1})} & 0 \end{pmatrix} \tag{8.1}$$

on $\tilde{\nu}_{2j}$, $j = 0, 1, \dots, N$, and, correspondingly,

$$\begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_j-\Omega_j)} \\ -e^{\frac{2i}{\varepsilon}(W_j-\Omega_j)} & 0 \end{pmatrix} \tag{8.2}$$

on $\tilde{\nu}_{2j-1}$, $j = 1, \dots, N$. In order to normalize this RHP, we introduce

$$\tilde{m}^{(\mathrm{mod})} = e^{-\frac{i}{\varepsilon}\Omega_1\sigma_3} m^{(\mathrm{mod})} e^{\frac{i}{\varepsilon}\Omega_1\sigma_3}.$$

Then the RHP $\widetilde{P}^{(\mathrm{mod})}$ for the matrix $\tilde{m}^{(\mathrm{mod})}$ becomes

$$\text{RHP } \tilde{P}^{(\mathrm{mod})}: \quad \tilde{m}_+^{(\mathrm{mod})} = \tilde{m}_-^{(\mathrm{mod})} \tilde{V}^{(\mathrm{mod})} \quad \text{when } z \in \tilde{\Sigma}^{(\mathrm{mod})}, \tag{8.3}$$

where the contour $\tilde{\Sigma}^{(\mathrm{mod})} = \bigcup_{j=0}^{2N} \tilde{\nu}_j$ and the piecewise constant jump matrix

$$\tilde{V}^{(\mathrm{mod})} = \begin{cases} \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_j-\Omega_{j+1}+\Omega_1)} \\ -e^{\frac{2i}{\varepsilon}(W_j-\Omega_{j+1}+\Omega_1)} & 0 \end{pmatrix} \\ \quad \text{when } z \in \tilde{\nu}_{2j},\ j = 0, \dots, N, \\ \begin{pmatrix} 0 & e^{-\frac{2i}{\varepsilon}(W_j-\Omega_j+\Omega_1)} \\ -e^{\frac{2i}{\varepsilon}(W_j-\Omega_j+\Omega_1)} & 0 \end{pmatrix} \\ \quad \text{when } z \in \tilde{\nu}_{2j-1},\ j = 1, \dots, N, \end{cases} \tag{8.4}$$

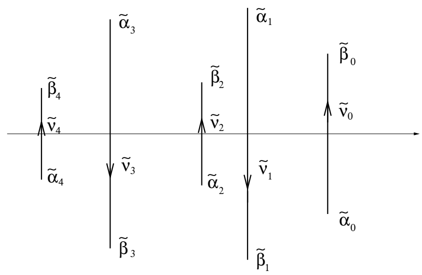


FIGURE 8.2. Contour $\tilde\Sigma^{(\mathrm{mod})}$.

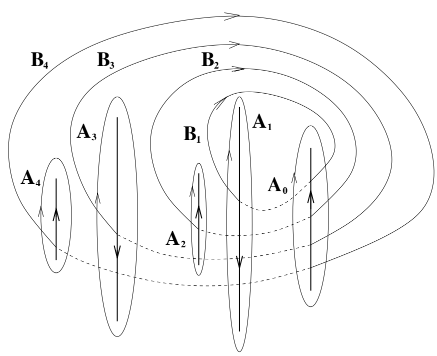


FIGURE 8.3. Basic cycles $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$.

and $\tilde m^{(\mathrm{mod})}(\infty) = e^{(2i/\varepsilon)g(\infty)\sigma_3}$; see Figure 8.2. It is now clear that $\tilde V^{(\mathrm{mod})} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\tilde\nu_0$.

To simplify notation, we write the piecewise constant matrix

$$\tilde V^{(\mathrm{mod})} = \begin{pmatrix} 0 & e^{i\tilde\Omega} \\ -e^{-i\tilde\Omega} & 0 \end{pmatrix}, \tag{8.5}$$

where $\tilde\Omega$ attains the corresponding values on each vertical segment $\tilde\nu_j$, $j = 0, 1, \ldots, 2N$. We also denote by $\tilde\alpha_j$ and $\tilde\beta_j$ the beginning and endpoints of the segment $\tilde\nu_j$ (so that $\tilde\alpha_0 = \alpha_1$, $\tilde\beta_0 = \alpha_0$, $\tilde\alpha_1 = \alpha_2, \ldots, \tilde\beta_{2N} = \alpha_{4N+1}$); see Figure 8.2.

As in Section 7, we introduce the canonical homology basis $\boldsymbol{A}_j, \boldsymbol{B}_j$, $j = 1, \ldots, N$, of the hyperelliptic surface $\tilde{\mathcal{R}}(x,t)$, determined by the cuts $\tilde\nu_j$ of RHP $\widetilde{P}^{(\mathrm{mod})}$; see Figure 8.3. The dotted curves in Figure 8.3 are passing through the second sheet. Some notation from Section 7, like $\omega, \theta$, etc., are used in the present section with respect to the hyperelliptic surface $\tilde{\mathcal{R}}(x,t)$ and homology basis $\boldsymbol{A}_j, \boldsymbol{B}_j$. In particular, we introduce $u(z) = \int_{\tilde\alpha_0}^{z} \omega$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_{2N})$ is the basis of holomorphic differentials dual to $\boldsymbol{A}$ cycles, i.e.,

$$\int_{\boldsymbol{A}_j} \omega_k = \delta_{jk}\,, \tag{8.6}$$

$j, k = 1, \dots, N$, and vector

$$\mathcal{M}(z,d) \equiv (\mathcal{M}_1, \mathcal{M}_2) = \left( \frac{\theta(u(z) - \frac{\hat\Omega}{2\pi} + d)}{\theta(u(z)+d)}, \frac{\theta(-u(z) - \frac{\hat\Omega}{2\pi} + d)}{\theta(-u(z)+d)} \right), \tag{8.7}$$

where $\hat\Omega = (\tilde\Omega_1, \tilde\Omega_2, \dots, \tilde\Omega_{2N})^{\mathsf{T}}$ and $d \in \mathbb{Z}^{2N}$ is a vector to be determined. Then $\mathcal{M}$ satisfies

$$\mathcal{M}_+ = \mathcal{M}_- \begin{pmatrix} 0 & e^{i\tilde\Omega} \\ e^{-i\tilde\Omega} & 0 \end{pmatrix} \tag{8.8}$$

on $\tilde\Sigma^{(\mathrm{mod})}$ (see [41]).

Introduce

$$\lambda(z) = \left( \prod_{j=0}^{2N} \frac{z - \tilde\beta_j}{z - \tilde\alpha_j} \right)^{\frac14} \tag{8.9}$$

with branch cuts along $\tilde\nu_j$. We choose the branch of $\lambda(z)$ such that $\lim_{z\to\infty} \lambda(z) = 1$ and that $\lambda_+ = i\lambda_-$ on $\tilde\nu_j$. One can verify directly that

$$\mathcal{L}_+ = \mathcal{L}_- \tilde V^{(\mathrm{mod})}\,, \tag{8.10}$$

where

$$\mathcal{L}(z) = \frac12 \begin{pmatrix} (\lambda(z) + \lambda^{-1}(z))\mathcal{M}_1(z,d) & -i(\lambda(z) - \lambda^{-1}(z))\mathcal{M}_2(z,d) \\ i(\lambda(z) - \lambda^{-1}(z))\mathcal{M}_1(z,-d) & (\lambda(z) + \lambda^{-1}(z))\mathcal{M}_2(z,-d) \end{pmatrix}. \tag{8.11}$$

If $\lambda(z) - \lambda^{-1}(z)$ has precisely $2N$ simple zeros $z_1, z_2, \dots, z_{2N}$, we choose

$$d = -\sum_{j=1}^{N} \int_{\tilde\alpha_j}^{X_2(z_j)} \omega_j\,, \tag{8.12}$$

where $X_2(z_j)$ is the preimage of $z_j$ on the second sheet of the hyperelliptic surface. This choice of $d$ implies that $\mathcal{L}(z)$ is analytic off $\tilde\Sigma^{(\mathrm{mod})}$. So,

$$\tilde m^{(\mathrm{mod})}(z) = e^{\frac{2i}{\varepsilon} g(\infty)\sigma_3} \mathcal{L}^{-1}(\infty)\mathcal{L}(z) \tag{8.13}$$

is the solution to the RHP $\tilde V^{(\mathrm{mod})}$ and

$$m^{(\mathrm{mod})}(z) = e^{\frac{2i}{\varepsilon}[g(\infty) + \frac12 \Omega_1]\sigma_3} \mathcal{L}^{-1}(\infty)\mathcal{L}(z) e^{-\frac{i}{\varepsilon}\Omega_1\sigma_3}\,. \tag{8.14}$$

Using now the fact that

$$\lambda(z) - \lambda^{-1}(z) = \frac{1}{2z}\sum_{j=0}^{2N}(\tilde{\alpha}_j - \tilde{\beta}_j) + O(z^{-2}) \quad \text{as } z \to \infty \tag{8.15}$$

and

$$m^{(2)} \to m^{(\text{mod})} e^{-\frac{2i}{\varepsilon} g(\infty)\sigma_3} \quad \text{as } \varepsilon \to 0\,,$$

we can rewrite (7.30) as

$$\begin{aligned} q_0(x,t,\varepsilon) &= -2\big(m_1^{(2)}\big)_{12} \\ &= \left(\frac{i}{2}\right)\frac{\mathcal{M}_2(\infty,d)}{\mathcal{M}_1(\infty,d)}\sum_{j=0}^{2N}(\tilde{\alpha}_j - \tilde{\beta}_j)e^{\frac{2i}{\varepsilon}[2g(\infty)+\Omega_1]} \\ &= \frac{\theta(u(\infty)+\frac{\hat{\Omega}}{2\pi}-d)\theta(u(\infty)+d)}{\theta(u(\infty)-\frac{\hat{\Omega}}{2\pi}+d)\theta(u(\infty)-d)} e^{\frac{2i}{\varepsilon}[2g(\infty)+\Omega_1]}\sum_{j=0}^{2N}(-1)^j b_j\,, \end{aligned} \tag{8.16}$$

where, according to (3.17),

$$g(\infty) = \frac{1}{2\pi i}\int_{\gamma_m}\frac{f(\zeta)+W}{R_+(\zeta)}\zeta^{2N}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma_c}\frac{\Omega}{R(\zeta)}\zeta^{2N}\,d\zeta\,. \tag{8.17}$$

We can now restate Theorem 7.1 as follows:

THEOREM 8.1 *Under the condition that* $\lambda(z) - \lambda^{-1}(z)$*, defined through* (8.9)*, has* $2N$ *simple zeros, the leading-order term of the solution to* (1.1)–(1.2) (*as* $\varepsilon \to 0$) *has the form* (8.16) *in the region of genus* $2N$.

The expression (8.16) for $q_0(x,t,\varepsilon)$ looks somewhat simpler than (7.29); however, it is worth mentioning that the constant vector $d$ in (8.16) requires additional calculations, and, most importantly, it is not clear how to check the condition about zeros of $\lambda(z) - \lambda^{-1}(z)$. Another advantage of expression (8.16) is that the constant vector $\hat{\Omega}$ in the argument of theta functions (which grows like $O(\varepsilon^{-1})$ as $\varepsilon \to \infty$) is real. Because of that latter fact, we calculate below $m^{(\text{mod})}$ through the RHP $\widetilde{P}^{(\text{mod})}$ exactly as in Section 7 (i.e., without any conditions on $\lambda(z)$) and get the following version of Theorem 7.1:

THEOREM 8.2 *In the region of genus* $2N$*, the leading-order term of the solution to* (1.1)–(1.2) (*as* $\varepsilon \to 0$) *has the form*

$$q_0(x,t,\varepsilon) = -2Ae^{\frac{2i}{\varepsilon}[2g(\infty)+\Omega_1]}\,, \tag{8.18}$$

*where* $g(\infty)$ *is given by* (8.17) *and*

$$
\begin{aligned}
A &= \frac{\mathcal{M}_2(\infty, d_1)\mathcal{M}_2(\infty, 0)}{\det \mathcal{N}(\infty)} \\
&\quad \times \left[ -\Im \sum_{j=0}^{2N} (\tilde{\beta}_j - \tilde{\alpha}_j) + i \nabla \ln \frac{\mathcal{M}_2(u(\infty), d_1)}{\mathcal{M}_2(u(\infty), 0)} \cdot \omega^0 \right] \\
&= \frac{\theta(0)\theta(d_1)}{\theta(u(\infty))\theta(u(\infty) + d_1)} \, \frac{\theta(u(\infty) + \frac{\hat{\Omega}}{2\pi} + d_1)\theta(u(\infty) + \frac{\hat{\Omega}}{2\pi})}{\theta(\frac{\hat{\Omega}}{2\pi} + d_1)\theta(\frac{\hat{\Omega}}{2\pi})} \\
&\quad \times \left[ -\sum_{j=0}^{2N} (-1)^j b_j + i \nabla \ln \frac{\theta(u(\infty) + \frac{\hat{\Omega}}{2\pi} + d_1)\theta(u(\infty))}{\theta(u(\infty) + d_1)\theta(u(\infty) + \frac{\hat{\Omega}}{2\pi})} \cdot \omega^0 \right].
\end{aligned} \tag{8.19}
$$

*Here theta functions, differentials* $\omega$*, and* $\mathcal{M}(z, d)$ *are defined through the hyperelliptic surface* $\tilde{\mathcal{R}}(x, t)$ *associated with the contour* $\tilde{\Sigma}^{(\mathrm{mod})}$*, i.e., through the vertical cuts* $\tilde{\nu}_j$*,* $j = 0, 1, \dots, 2N$*, and the canonical homology basis* $\boldsymbol{A}_j$*,* $\boldsymbol{B}_j$*,* $j = 1, \dots, N$; *see Figure* 8.3.

## 9 Proof of the Error Estimate (1.12) in the Main Theorem (Theorem 1.1)

To complete the proof of the theorem, it remains to construct the parametrix at $\frac{\mu}{2}$ and establish (2.37). In our derivation of the parametrix near $\frac{\mu}{2}$ and of the error estimate (2.37), we will make repeated use of some general formulae, derived below. To estimate errors in an RHP $m_+ = m_- v$, $m \to I$ as $z \to \infty$ over a contour $\Gamma$, one needs to consider certain singular integral equations for RHPs. One of the equations we may use is

$$m_- = I + C_\Gamma^- m_-(v - I)\,, \tag{9.1}$$

where the Cauchy integral operator $C_\Gamma$ is as described in Section 4.2. Once this equation is solved for $m_-$, we have

$$m = I + C_\Gamma m_-(v - I) \equiv I + C_v m_-\,. \tag{9.2}$$

It is easily checked that $m_+ = m_- v$ by using the identity $C_\Gamma^+ - C_\Gamma^- = I$; see [41]. To obtain estimates with respect to external parameters, in most cases we need to have the bound

$$\left\| (1 - C_v)^{-1} \right\|_{L^2 \circlearrowleft} \le c \tag{9.3}$$

uniformly with respect to the external parameters involved. Here $\mathcal{B} \circlearrowleft$ denotes the Banach algebra of the bounded operators acting on the Banach space $\mathcal{B}$. We will use the formula

$$m_- - I = (I - C_v)^{-1} C_v I\,, \tag{9.4}$$

which is easily derived from (9.1).

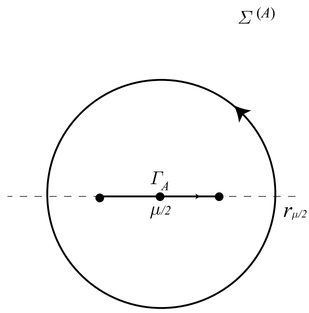


FIGURE 9.1. RHP $P^{(A)}$.

**Construction of the Parametrix near $\frac{\mu}{2}$**

In order to construct $m^{(\mathrm{app})}$ near $\frac{\mu}{2}$, we set $m^{(\mathrm{app})} = A(z)m^{(\mathrm{mod})}(z)$, where a $2 \times 2$ matrix function $A(z)$ satisfies the RHP $P^{(A)}$:

$$(9.5)\qquad \begin{aligned} (A(z)m^{(\mathrm{mod})}(z))_+ &= (A(z)m^{(\mathrm{mod})}(z))_- V^{(4)}(z)\,, \quad z \in \Gamma_A\,, \\ A(z) &\to I \quad \text{as } z \to \infty\,; \end{aligned}$$

see Figure 9.1. Here $\Gamma_A = [\frac{\mu}{2} - \delta, \frac{\mu}{2} + \delta]$.

Since $m^{(\mathrm{mod})}(z)$ is analytic near $\frac{\mu}{2}$, the jump condition of $A(z)$ becomes

$$(9.6)\quad A(z)_+ = A(z)_- v_A \quad \text{where } v_A = m^{(\mathrm{mod})}(z) V^{(4)}(z) m^{(\mathrm{mod})}(z)^{-1}\,, \ \ z \in \Gamma_A\,.$$

The jump matrix $V^{(4)}$ and hence also the jump matrix $v_A$ have the form

$$(9.7)\qquad V^{(4)}, v_A = I + O(e^{-c|z-\frac{\mu}{2}|/\varepsilon}) \quad \text{as } \varepsilon \to 0$$

for some $c > 0$ uniformly in a neighborhood of $\frac{\mu}{2}$ that does not depend on $\varepsilon$. Equation (9.7) follows from (2.15)–(2.16) and (4.66)–(4.67).

THEOREM 9.1 *There exists a solution $A(z)$ to the RHP $P^{(A)}$ satisfying the estimate $\|A - I\|_{L^\infty(r_{\mu/2})} = O(\varepsilon)$, where $r_{\mu/2}$ is a circle centered at $\frac{\mu}{2}$ with radius $2\delta$.*

PROOF: Equation (9.1) applied to the RHP $P^{(A)}$ becomes

$$(9.8)\qquad A_- = I + C^-_{\Gamma_A} A_-(v_A - I)\,.$$

Once this equation is solved for $A_-$ (we show existence below), (9.2) becomes

$$(9.9)\qquad A = I + C_{\Gamma_A} A_-(v_A - I) \equiv I + C_v A_-\,.$$

We retain the notation $C_v$ instead of the more consistent $C_{v_A}$ to avoid more sub-indexing.

One sees directly from Schwarz symmetry that the jump matrix $v_A(z)$ is positive definite for $z \in \Gamma_A$. Indeed, in (9.6) $V^{(4)}$ is positive definite and $m^{(\mathrm{mod})}(z)$ is unitary on $\Gamma_A$; the latter follows from the fact that $V^{(\mathrm{mod})}$ is unitary on $\Sigma^{(\mathrm{mod})}$ and the RHP $P^{(\mathrm{mod})}$ has a unique solution. By [41], we have the a priori bound

$$\left\|(1-C_v)^{-1}\right\|_{L^2(\Gamma_A)\circlearrowleft} \leq \frac{\lambda_{\max}+1+\sqrt{(\lambda_{\max}+1)^2-4\lambda_{\min}}}{2\lambda_{\min}}, \tag{9.10}$$

where

$$\lambda_{\min} = \operatorname*{ess\,inf}_{z\in\Gamma_A}\{\text{minimal eigenvalue of } v(z)\}$$

and

$$\lambda_{\max} = \operatorname*{ess\,sup}_{z\in\Gamma_A}\{\text{maximal eigenvalue of } v(z)\}.$$

Thus we only need to show that

$$\left\|v_A^{\pm 1}\right\|_{L^\infty} \leq c_1 \tag{9.11}$$

for some $c_1$ independent of $\varepsilon$ and also independent of $x$, $t$, and $\mu$, etc.

Since $m^{(\mathrm{mod})}(z)$ is unitary for $z\in\Gamma_A$, (9.11) is equivalent to

$$\left\|v^{\pm 1}\big|_{\Gamma_A}\right\|_{L^\infty} \leq c_2 \tag{9.12}$$

for some $c_2 > 0$. But this is obvious from the expression of $v_A$. This proves the existence of $A(z)$.

We now estimate $A(z)$ on the circle $r_{\mu/2}$. According to (9.9),

$$A(z) = I + \frac{1}{2\pi i}\int\limits_{\Gamma_A}\frac{A_-(\zeta)(v_A(\zeta)-I)d\zeta}{\zeta-z}, \quad z\in r_{\mu/2}. \tag{9.13}$$

Since $|\zeta - z| \geq \delta$,

$$\|A-I\|_{L^\infty(\gamma_A)} \leq \frac{1}{2\pi\delta}\left(\|A_--I\|_{L^2(r_{\mu/2})}\|v_A-I\|_{L^2(\Gamma_A)} + \|v_A-I\|_{L^1(\Gamma_A)}\right) \tag{9.14}$$

for some $c$. The form (9.6) of $v_A$ yields

$$\|v_A-I\|_{L^2(\Gamma_A)} = O(\varepsilon^{1/2}), \quad \|v_A-I\|_{L^1(\Gamma_A)} = O(\varepsilon). \tag{9.15}$$

Thus, using (9.4),

$$\begin{aligned}\|A_--I\|_{L^2(\Gamma_A)} &\leq \|(1-C_v)^{-1}\|_{L^2\circlearrowleft}\|C_{v_A}I\|_{L^2(\Gamma_A)}\\ &\leq \|V_A-I\|_{L^2(\Gamma_A)} = O(\varepsilon^{1/2}).\end{aligned}$$

Finally, by (9.14),

$$\|A-I\|_{L^\infty(r_{\mu/2})} = O(\varepsilon). \tag{9.16}$$

This is the required estimate. □

PROOF OF THE MAIN ESTIMATE (2.37): The introduction of $m^{(\mathrm{err})}$ peels off the matrix $m^{(\mathrm{app})}$ from the RHP $P^{(4)}$. The matrix $m^{(\mathrm{err})}$ solves the RHP $P^{(\mathrm{err})}$ that has a jump uniformly close to $I$ as $\varepsilon\to 0$ on $\Sigma^{(\mathrm{err})}$; see Figure 2.11. Therefore [41]

$$\left\|(1-C_{V^{(5)}})^{-1}\right\|_{L^2\circlearrowleft} \leq c_3 \tag{9.17}$$

for some $c_3$ and for $\varepsilon$ small enough. In the complement of the three circles $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $|z - \frac{\mu}{2}| = \delta$ centered at $\alpha_0$, $\bar{\alpha}_0$, and $\frac{\mu}{2}$, respectively, we obtain

$$\|V^{(\mathrm{err})} - I\|_{L^p} \le c_4 e^{-c_5/\varepsilon}\,, \quad 1 \le p \le \infty, \text{ for some } c_4, c_5 > 0\,.$$

On the circles $r_{\alpha_0}$, $r_{\bar{\alpha}_0}$, and $r_{\mu/2}$, we have

$$\|V^{(\mathrm{err})} - I\|_{L^p} = O(\varepsilon)\,, \quad 1 \le p \le \infty\,. \tag{9.18}$$

Using (9.4), we obtain

$$\begin{aligned}\left\|m_-^{(\mathrm{err})} - I\right\|_{L^2} &\le \left\|(1 - C_{V^{(\mathrm{err})}})^{-1}\right\|_{L^2 \circlearrowleft} \|C_{V^{(\mathrm{err})}} I\|_{L^2} \\ &\le c_6 \|V^{(\mathrm{err})} - I\|_{L^p} = O(\varepsilon)\end{aligned} \tag{9.19}$$

for some $c_6 > 0$.

Now we are ready to estimate $(m_1^{(\mathrm{err})} - I)_{12}$, the (12)-entry of the matrix $m_1^{(\mathrm{err})} - I$ defined by the expansion which, as mentioned earlier, is the error term. Using (2.37) and (9.2), we obtain

$$m_1^{(\mathrm{err})} = -\frac{1}{2\pi i} \int_{\Sigma^{(\mathrm{err})}} m_-^{(\mathrm{err})}(\zeta)(V^{(\mathrm{err})}(\zeta) - I)d\zeta\,.$$

Therefore,

$$\left|m_1^{(\mathrm{err})}\right| \le c\big(\|m_-^{(\mathrm{err})} - I\|_{L^2}\, \|V^{(\mathrm{err})} - I\|_{L^2} + \|V^{(\mathrm{err})} - I\|_{L^1}\big) = O(\varepsilon) \tag{9.20}$$

completes the proof of estimate (2.37) and, thus, of (1.12). □

## Appendix A: Riemann-Hilbert Formalism for the Inverse Scattering Problem

We consider a Zakharov-Shabat system

$$i\varepsilon W' = \begin{pmatrix} z & q \\ \bar{q} & -z \end{pmatrix} W\,, \tag{A.1}$$

where a function $q(x)$ bounded on $\mathbb{R}$ is called a *potential* and $z \in \mathbb{C}$ is called a *spectral parameter*. This system has the following symmetry $R$: $W$ is a solution for (A.1) with some $z \in \mathbb{C}$ if and only if $\widehat{W} = i\sigma_2 \overline{W}$ is a solution for (A.1) with $\bar{z}$. This fact can be easily verified. Here and henceforth

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

denote the corresponding Pauli matrices.

Suppose now that the potential $q$ is a continuous function that approaches 0 as $x \to \pm\infty$. We introduce a pair of Jost solutions $\Psi$ and $\Phi$ by asymptotic conditions

$$\Psi_1 \sim \mathrm{Col}(1, 0)e^{\frac{-izx}{\varepsilon}}\,, \quad \Psi_2 \sim \mathrm{Col}(0, 1)e^{\frac{izx}{\varepsilon}}\,, \quad \text{as } x \to \infty,\ z \in \mathbb{R}\,, \tag{A.2}$$

and

$$\Phi_1 \sim \mathrm{Col}(1,0)e^{\frac{-izx}{\varepsilon}}, \quad \Phi_2 \sim \mathrm{Col}(0,1)e^{\frac{izx}{\varepsilon}}, \quad \text{as } x \to -\infty,\ z \in \mathbb{R}. \tag{A.3}$$

Jost solutions $\Psi$ and $\Phi$ are fundamental solutions to (A.1), which are uniquely defined by (A.2) and (A.3), respectively. Note that

$$\hat{\Phi}_1(x,z) = -\Phi_2(x,\bar{z}) \quad \text{and} \quad \hat{\Phi}_2(x,z) = \Phi_1(x,\bar{z})\,. \tag{A.4}$$

Similar relations take place for the Jost solution $\Psi$.

Let $\Phi_1 = a\Psi_1 + b\Psi_2$, where the coefficients $a$ and $b$ depend on $z$ and $\varepsilon$. Then

$$\frac{1}{a}\Phi_1(x,z) = \Psi_1(x,z) + \frac{b}{a}\Psi_2(x,z)\,. \tag{A.5}$$

The ratio $r = \frac{b}{a}$ is called a *reflection coefficient*, whereas $\frac{1}{a}$ is called a *transmission coefficient* in scattering theory. Roughly speaking, the objective of the inverse scattering problem is to reconstruct the potential $q$ through the reflection coefficient $r$. Note that the change $q \mapsto -q$ implies $r \mapsto -r$. Indeed, $q \mapsto -q$ implies $W \mapsto \sigma_3 W$ for an arbitrary solution $W$ to (A.1), so that $r \mapsto -r$ follows from (A.2), (A.3), and (A.5).

Applying to the latter equation the symmetry operation $R$ and taking into account (A.4), we obtain

$$\frac{1}{\bar{a}}\hat{\Phi}_2(x,\bar{z}) = -\bar{r}\Psi_1(x,\bar{z}) + \Psi_2(x,\bar{z})\,. \tag{A.6}$$

Consider now the matrices $M_+ = (\frac{\Phi_1}{a}, \Psi_2)$ and $M_- = (\Psi_1, \frac{\Phi_2}{a})$, defined for $z \in \mathbb{R}$ in the upper and lower half-planes, respectively. Then, for $z \in R$, we obtain $M_+ = M_- W$, where the so-called jump matrix $W$ is determined by

$$\begin{aligned} M_+ = \left(\frac{\Phi_1}{a}, \Psi_2\right) &= (\Psi_1 + r\Psi_2, \Psi_2) \\ &= (\Psi_1, \Psi_2 - \bar{r}\Psi_1)\begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix} = M_- W\,. \end{aligned}$$

Note that both $M_\pm$ are fundamental solutions of (A.1). It is easy to check that (1) the matrix $m = M\exp(-\frac{izx}{\varepsilon}\sigma_3)$ satisfies the differential equation

$$i\varepsilon\frac{d}{dx}m = z[\sigma_3, m] + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} m\,, \tag{A.7}$$

and (2) the jump matrix for $m$ on the line $\Im z = 0$ is

$$V = \begin{pmatrix} 1+|r|^2 & \bar{r}e^{-2\frac{izx}{\varepsilon}} \\ re^{2\frac{izx}{\varepsilon}} & 1 \end{pmatrix}. \tag{A.8}$$

The inverse scattering problem can be viewed now as the RH problem for the matrix function $m$, i.e., the problem of reconstructing the matrices $m_+$ and $m_-$, which are analytic and analytically invertible in $z$ in the upper and lower half-planes, respectively, that satisfy the jump condition with the jump matrix (A.8) on the real line and that are normalized by $\lim_{z\to\infty} m_\pm = I$, where $I$ denotes the identity matrix. Indeed, if $m$ is the solution to the above RH problem, then $m = I + \frac{m_1}{z} + O(z^{-2})$ is analytic at infinity. Substituting this expression into (A.7), we obtain

$$q = -2(m_1)_{12}\,, \tag{A.9}$$

where the notation $(m)_{ij}$ stands for the $(ij)$–entry of the matrix $m$.

It is well-known that if the potential $q_0(x,t,\varepsilon)$ evolves in $t$ according to the NLS of our interest, then the evolution of the reflection coefficient is given by $r(t) = r(0)\exp(\frac{4iz^2t}{\varepsilon^2})$. So, to reconstruct $q = q_0(x,t,\varepsilon)$, we have to solve the RH problem

$$m_+ = m_- \begin{pmatrix} 1+|\tilde{r}|^2 & \tilde{r}^* \\ \tilde{r} & 1 \end{pmatrix} = m_- V\,, \tag{A.10}$$

where $\tilde{r} = r\exp((\frac{4iz^2t}{\varepsilon^2} + 2\frac{izx}{\varepsilon})$, $r = r(0)$, and $\tilde{r}^*(z)$ denotes the function that is complex conjugate to $\tilde{r}(z)$ for $z \in \mathbb{R}$. The upper half-plane is the positive side. This is the original problem.

## Appendix B: Asymptotic Properties of the Reflection Coefficient

The singularities of the reflection $r^{(0)}(z)$ in the upper half-plane are simple poles located at the points $w_+ - w = k$ and $w - 1 - w_+ - w_- = j$, where $k$ and $j$ are nonnegative integers [33]. Taking into account (1.7), these equations lead to two strings of $\varepsilon$-spaced poles

$$z_k = i\left[\sqrt{1-\frac{\mu^2}{4}} - \varepsilon\Big(k+\frac{1}{2}\Big)\right], \quad z_j = \frac{\mu}{2} + i\varepsilon\Big(j+\frac{1}{2}\Big). \tag{B.1}$$

The string $z_j$ consists of "nonsoliton" poles. The string $z_k$ determines the standard, or "soliton," poles of the reflection coefficient. They are present in the upper half-plane only if $\mu < 2$.

In order to use the Stirling formula to obtain the asymptotic expansion of the reflection coefficient as $\varepsilon \to 0$, we have to exclude the regions on the complex upper half-plane, where the Stirling formula is not valid. It is easy to see that, in our case, there are two such regions $D_{1,2}$: the region $D_1$ is the union of a small half-disk around $\frac{\mu}{2}$, lying in the upper half-plane, and a narrow sector, centered at $\frac{\mu}{2}$, and containing all the poles $z_j$; the region $D_2$ (which is nonempty only in the case $\mu < 2$) is the union of a small disk around $T$ and the part of a narrow sector, centered at $T$, containing all the poles $z_k$ lying in the upper half-plane.

Let $S = S_1 \cup S_2$, where $S_1$ and $S_2$ are parts of the upper half-plane to the left and to the right of the region $D_1$, where $S_1$ does not contain $D_2$; see Figure B.1. It is easy to see that all the gamma functions in (1.6) have uniform Stirling asymptotic expansions in $S$.

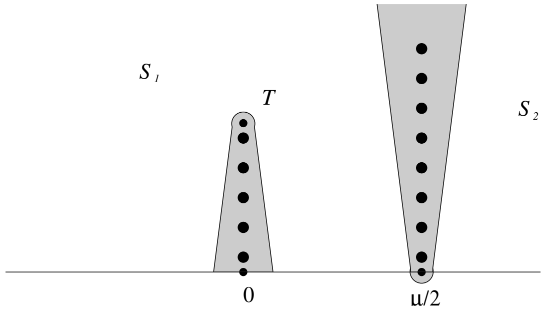


FIGURE B.1. Regions of Stirling asymptotics.

We assume that all the gamma functions in $r^{(0)}(z)$ are defined in $S_1$. All of them but $\Gamma(1-w+w_++w_-)$ have analytic continuation to the rest of $S$ through the ray $\Re z = \frac{\mu}{2}$, whereas $\Gamma(1-w+w_++w_-)$ has analytic continuation below this ray.

LEMMA B.1 *The function* $f(z,\varepsilon) = \frac{1}{2} i\varepsilon \ln r^{(0)}(z)$ *has asymptotics*

$$f(z,\varepsilon) = f_0^{(0)}(z) + \frac{\pi}{2}\varepsilon + f_2(z)\varepsilon^2 + O(\varepsilon^3)\,, \quad \varepsilon \to 0\,, \tag{B.2}$$

*uniformly in S, where*

$$\begin{aligned} f_0^{(0)}(z) &= \Big(\frac{\mu}{2} - z\Big)\Big[\frac{i\pi}{2} + \ln(\frac{\mu}{2} - z)\Big] + \frac{z+T}{2}\ln(z+T) \\ &\quad + \frac{z-T}{2}\ln(z-T) - T\tanh^{-1}\frac{T}{\mu/2} + \frac{\mu}{2}\ln 2 \quad \textit{if } z \in S_1, \\ f_0^{(0)}(z) &= \Big(\frac{\mu}{2} - z\Big)\Big[\frac{i\pi}{2} + \ln(z - \frac{\mu}{2})\Big] + \frac{z+T}{2}\ln(z+T) \\ &\quad + \frac{z-T}{2}\ln(z-T) - T\tanh^{-1}\frac{T}{\mu/2} + \frac{\mu}{2}\ln 2 \quad \textit{if } z \in S_2, \\ f_2^{(0)}(z) &= -\frac{\mu}{24}\Big[\frac{5z}{z^2 - T^2} + \frac{1}{z - \frac{\mu}{2}} - \mu\Big]. \end{aligned} \tag{B.3}$$

*Here we take the standard branches of all logarithms*; *i.e., the logarithms are real when their arguments are positive.*

PROOF: By definition,

$$
\begin{aligned}
f(z,\varepsilon) &= \frac{1}{2} i\varepsilon \ln r^{(0)}(z) \\
&= \frac{i\varepsilon}{2}\Bigg\{ -\frac{i\pi}{2} + \ln\varepsilon - \frac{i\mu}{\varepsilon}\ln 2 + \ln \frac{\Gamma(\frac{1}{2} + \frac{i}{\varepsilon}(z - \frac{\mu}{2}))}{\Gamma(\frac{1}{2} - \frac{i}{\varepsilon}(z - \frac{\mu}{2}))} \\
&\qquad + \ln\Big[\Gamma\Big(\frac{1}{2} - \frac{i}{\varepsilon}(z+T)\Big)\Gamma\Big(\frac{1}{2} - \frac{i}{\varepsilon}(z-T)\Big)\Big] \\
&\qquad - \ln\Big[\Gamma\Big(\frac{i}{\varepsilon}\Big(T - \frac{\mu}{2}\Big)\Big)\Gamma\Big(-\frac{i}{\varepsilon}\Big(T + \frac{\mu}{2}\Big)\Big)\Big]\Bigg\},
\end{aligned}
\tag{B.4}
$$

where $\Im z \geq 0$. To determine the asymptotics of $f$, we will make use of the Stirling formula

$$
\ln\Gamma(s) = \Big(s - \frac{1}{2}\Big)\ln s - s + \frac{1}{2}\ln(2\pi) + \frac{1}{12s} + O(s^{-2}) \qquad \text{as } s \to \infty,\ |\arg s| < \pi\,,
\tag{B.5}
$$

which is valid in any sector $|\arg s| < \pi$ of the complex $s$-plane.

Let us denote as $\ln r_j$, $j = 1, 2, 3$, the last three logarithmic terms in (B.4). Direct application of the Stirling formula to $\ln r_j$, $j = 1, 2, 3$, yields

$$
\begin{aligned}
\ln r_1 &\sim \frac{i}{\varepsilon}\Big(z - \frac{\mu}{2}\Big)\Big[\ln\Big(\frac{1}{2} + \frac{i}{\varepsilon}\Big(z - \frac{\mu}{2}\Big)\Big) + \ln\Big(\frac{1}{2} - \frac{i}{\varepsilon}\Big(z - \frac{\mu}{2}\Big)\Big) - 2\Big] \\
&\quad - \frac{i\varepsilon}{6(z - \frac{\mu}{2})}, \\
\ln r_2 &\sim -\frac{i}{\varepsilon}(z+T)\Big[\ln\Big(-\frac{i}{\varepsilon}(z+T)\Big) + \ln\Big(1 + \frac{i\varepsilon}{2(z+T)}\Big)\Big] \\
&\quad - \frac{i}{\varepsilon}(z-T)\Big[\ln\Big(-\frac{i}{\varepsilon}(z-T)\Big) + \ln\Big(1 + \frac{i\varepsilon}{2(z-T)}\Big)\Big] \\
&\quad + 2\frac{iz}{\varepsilon} - 1 + \ln(2\pi) + \frac{i\varepsilon z}{6(z^2 - T^2)} \\
-\ln r_3 &\sim \frac{i}{\varepsilon}\Big(\frac{\mu}{2} + T\Big)\ln\Big(-\frac{i}{\varepsilon}\Big(\frac{\mu}{2} + T\Big)\Big) \\
&\quad + \frac{i}{\varepsilon}\Big(\frac{\mu}{2} - T\Big)\ln\Big(-\frac{i}{\varepsilon}\Big(\frac{\mu}{2} - T\Big)\Big) \\
&\quad + \frac{1}{2}\ln\Big(\frac{\mu^2}{4} - T^2\Big) + \ln\Big(-\frac{i}{\varepsilon}\Big) - \frac{i\mu}{\varepsilon} - \frac{i\varepsilon\mu}{12}
\end{aligned}
\tag{B.6}
$$

with the accuracy of $O(\varepsilon^2)$.

It is now easy to check that $\ln r_2$ and $\ln r_3$ have, correspondingly, 0 and $-\frac{i\pi}{2}$ constant terms in the $\varepsilon$ expansion. Substituting (B.6) into (B.4), we see after some algebra that all terms containing $\ln\varepsilon$ cancel each other and we obtain

$$\begin{aligned} f(z,\varepsilon) = {} & \left(\frac{\mu}{2}-z\right)\left[\frac{i\pi}{2}+\ln\left(\frac{\mu}{2}-z\right)\right]+\frac{z+T}{2}\ln(z+T) \\ & +\frac{z-T}{2}\ln(z-T)-T\tanh^{-1}\frac{T}{\mu/2}-xz-2tz^2 \\ & +\frac{\mu}{2}\ln 2+\frac{\pi}{2}\varepsilon-\frac{\varepsilon^2\mu}{24}\left[\frac{5z}{z^2-T^2}+\frac{1}{z-\frac{\mu}{2}}-\mu\right]+O(\varepsilon^3) \end{aligned} \tag{B.7}$$

as $\varepsilon \to 0$ uniformly in $S$, where the $O(\varepsilon^3)$ part has asymptotic expansion in powers of $\varepsilon$ starting with $\varepsilon^3$. The proof is complete. □

COROLLARY B.2 *On the real z-axis, $|r^{(0)}(z)|$ is exponentially growing on $(-\frac{\mu}{2},\frac{\mu}{2})$ and exponentially decaying on $(-\infty,-\frac{\mu}{2})\cup(\frac{\mu}{2},\infty)$ as $\varepsilon\to 0$. The growth and decay are uniform away from some neighborhoods of the points $\pm\frac{\mu}{2}$.*

PROOF: Since $|r^{(0)}(z)| = e^{-2\frac{i}{\varepsilon}\Im f(z,\varepsilon)}$, the growth or decay of $|r^{(0)}(z)|$ is determined by the sign of $\Im f(z,\varepsilon)$. According to (B.2),

$$\Im f(z,\varepsilon) = \left(\frac{\mu}{2}-z\right)\left[\frac{i\pi}{2}+\Im\ln\left(\frac{\mu}{2}-z\right)\right]+\frac{z}{2}\Im\ln(z^2-T^2)+O(\varepsilon^3) \tag{B.8}$$

when $z\in\mathbb{R}$. Note that $\Im\ln(z^2-T^2)=0$ if $z\ge T$ when $\mu\ge 2$ and if $z>0$ when $\mu<2$, and $\Im\ln(z^2-T^2)=2\pi$ if $z\le -T$ when $\mu\ge 2$ and if $z<0$ when $\mu<2$. If $z\in(-T,T)$, then $\Im\ln(z^2-T^2)=-\pi$ when $\mu\ge 2$. According to the remark about the analytic continuation of gamma functions, the $\ln(\frac{\mu}{2}-z)$ term in (B.2) is real for all real $z\ne\frac{\mu}{2}$. Combining these observations, we get $\Im f_0(z)=(\frac{\mu}{2}-z)\frac{\pi}{2}$ if $z\ge T$ when $\mu\ge 2$ and if $z>0$ when $\mu<2$; $\Im f_0(z)=(\frac{\mu}{2}-z)\frac{\pi}{2}+\pi z=(\frac{\mu}{2}+z)\frac{\pi}{2}$ if $z\le -T$ when $\mu\ge 2$ and if $z<0$ when $\mu<2$. If $z\in(-T,T)$ then $\Im f_0(z)=(\frac{\mu}{2}-z)\frac{\pi}{2}+\frac{\pi}{2}z=\frac{\mu}{2}\frac{\pi}{2}$ when $\mu\ge 2$. As we see, in all cases $\Im f_0(z)>0$ on $(-\frac{\mu}{2},\frac{\mu}{2})$ and negative if $|z|>\frac{\mu}{2}$, $z\in\mathbb{R}$. Thus, $|r^{(0)}(z)|$ is exponentially growing on $(-\frac{\mu}{2},\frac{\mu}{2})$ and exponentially decreasing outside $[-\frac{\mu}{2},\frac{\mu}{2}]$. The corollary is proven. □

*Remark* B.3. It is easy to see that

$$|r(z,\varepsilon)| = \varepsilon\left|\frac{\Gamma(\frac{1}{2}-\frac{i}{\varepsilon}(z+T))\Gamma(\frac{1}{2}-\frac{i}{\varepsilon}(z-T))}{\Gamma(\frac{i}{\varepsilon}(T-\frac{\mu}{2}))\Gamma(-\frac{i}{\varepsilon}(T+\frac{\mu}{2}))}\right|. \tag{B.9}$$

In the case $|\mu|\ge 2$, i.e., in the no-solitons case, there is a simple way to analyze the behavior of $|r^{(0)}(z)|$ on $\mathbb{R}$. Indeed, we can use the well-known properties of the gamma function (see, e.g., section 8.332 in [21]) to evaluate

$$|r(z,\varepsilon)| = \left|\frac{\cosh(\frac{2\pi T}{\varepsilon})-\cosh(\pi\frac{\mu}{\varepsilon})}{\cosh(\frac{2\pi T}{\varepsilon})+\cosh(2\pi\frac{z}{\varepsilon})}\right|^{\frac{1}{2}}. \tag{B.10}$$

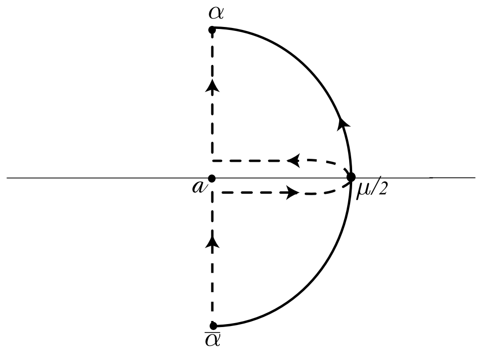


FIGURE C.1. Deformation of the contour.

According to this estimate, $|r(z,\varepsilon)|$ grows exponentially in $\frac{1}{\varepsilon}$ as $\varepsilon \to 0^+$ when $|z| < |\frac{\mu}{2}|$ and decreases exponentially when $|z| > |\frac{\mu}{2}|$.

*Remark* B.4. The conclusion of the lemma is also valid for $\mu = 0$. It is interesting, however, to consider the behavior of $r^{(0)}$ at $z = 0$. In the case $\mu = 0$ we have $\alpha = -\beta = \frac{1}{\varepsilon}$, $\gamma = \frac{1}{2} - \frac{iz}{\varepsilon}$, so that

$$r(0,\varepsilon) = -i\varepsilon \frac{\Gamma(\frac{1}{2} - \frac{1}{\varepsilon})\Gamma(\frac{1}{2} + \frac{1}{\varepsilon})}{\Gamma(-\frac{1}{\varepsilon})\Gamma(\frac{1}{\varepsilon})} = \tanh \frac{i\pi}{\varepsilon}. \tag{B.11}$$

As we see, $r(0,\varepsilon)$ is singular when $\varepsilon = \frac{1}{k+1/2}$, $k \in \mathbb{N}$.

## Appendix C: Calculations of Equations (4.2)

To compute the integrals (4.2) we deform the contour of integration as shown on Figure C.1.

If $f(\bar{z}) = \overline{f(z)}$ for all $z$, then both $f'(z)$ and $zf'(z)$ possess this property. Thus, for any such function $f$,

$$\begin{aligned}
&\int_{\overline{\alpha}}^{\alpha} \frac{f(\zeta)}{R_+(\zeta)}\,d\zeta + \int_{\Re\alpha}^{\frac{\mu}{2}} \frac{\overline{f}(\zeta) - f(\zeta)}{R_+(\zeta)}\,d\zeta \\
&= 2i\left[\int_0^b \Re f(a+i\beta)\frac{d\beta}{\sqrt{b^2-\beta^2}} - \int_a^{\frac{\mu}{2}} \frac{\Im[f(x)]dx}{\sqrt{b^2+(x-a)^2}}\right] \\
&= 0
\end{aligned} \tag{C.1}$$

where $\alpha = a + ib$ and $\zeta = a + i\beta$ for the first integral. Therefore, the system (4.2) can be written as

$$\int_0^b \Re\varphi(a+i\beta)\frac{d\beta}{\sqrt{b^2-\beta^2}} - \int_0^{\frac{\mu}{2}-a} \Im\varphi(a+\xi)\frac{d\xi}{\sqrt{b^2+\xi^2}} = 0\,, \tag{C.2}$$

where the function $\varphi$ is substituted for both $f'(z)$ and $zf'(z)$.

Note that

$$\Im f'(a+\xi) = \frac{\pi}{2}, \quad \Im(a+\xi) f'(a+\xi) = \frac{\pi}{2}(a+\xi). \tag{C.3}$$

Since $\frac{1}{2}\ln(z^2+1-\frac{\mu^2}{4}) = \frac{1}{2}[\ln(z-T)+\ln(z+T)]$, we obtain

$$\begin{aligned}
\Re f'(a+i\beta) &= \frac{1}{2}\big(\ln\left[(a-T)^2+\beta^2\right] + \ln\left[(a+T)^2+\beta^2\right]\big) \\
&\quad - \frac{1}{2}\ln\left[\left(\frac{\mu}{2}-a\right)^2+\beta^2)\right] - 4ta - x,
\end{aligned}$$

$$\begin{aligned}
\Re(a+i\beta) f'(a+i\beta) &= \frac{a}{4}\big(\ln\left[(a-T)^2+\beta^2\right] + \ln\left[(a+T)^2+\beta^2\right]\big) \\
&\quad - \frac{\beta}{2}\left[\tan^{-1}\frac{\beta}{a-T} + \tan^{-1}\frac{\beta}{a+T}\right] \\
&\quad - \frac{a}{2}\ln((1-a)^2+\beta^2) \\
&\quad - \beta\tan^{-1}\frac{\beta}{\frac{\mu}{2}-a} + \frac{\pi\beta}{2} - 4t(a^2-\beta^2) - ax,
\end{aligned} \tag{C.4}$$

where the fact that $\Re[z\ln z] = \Re z \ln|z| - \Im z \arg z$ is used for the latter expression.

Computations of the integrals of (C.3) yield

$$\begin{aligned}
-\int_0^{\frac{\mu}{2}-a} \frac{\Im[f'(a+\xi)]d\xi}{\sqrt{b^2+\xi^2}} &= \frac{\pi}{2}\int_0^{\frac{\mu}{2}-a} \frac{d\xi}{\sqrt{b^2+\xi^2}} \\
&= \frac{\pi}{2}\ln\left(\xi+\sqrt{\xi^2+b^2}\right)\Big|_0^{\frac{\mu}{2}-a} \\
&= \frac{\pi}{2}\left[\ln\left(\frac{\mu}{2}-a+\sqrt{\left(\frac{\mu}{2}-a\right)^2+b^2}\right) - \ln b\right]
\end{aligned} \tag{C.5}$$

and

$$\begin{aligned}
&-\int_0^{\frac{\mu}{2}-a} \frac{\Im[(a+\xi)f'(a+\xi)]d\xi}{\sqrt{b^2+\xi^2}} \\
&\quad = \frac{\pi}{2}\int_0^{\frac{\mu}{2}-a} \frac{(a+\xi)d\xi}{\sqrt{b^2+\xi^2}} \\
&\quad = \frac{\pi}{2}\left[a\int_0^{\frac{\mu}{2}-a} \frac{d\xi}{\sqrt{b^2+\xi^2}} + \int_0^{\frac{\mu}{2}-a} \frac{\xi d\xi}{\sqrt{b^2+\xi^2}}\right] \\
&\quad = \frac{\pi}{2}\left[\sqrt{b^2+\xi^2} + a\ln(\xi+\sqrt{\xi^2+b^2})\right]_0^{\frac{\mu}{2}-a} \\
&\quad = \frac{\pi}{2}\Bigg[\sqrt{\left(\frac{\mu}{2}-a\right)^2+b^2} \\
&\qquad + a\ln\left(\left(\frac{\mu}{2}-a\right)+\sqrt{\left(\frac{\mu}{2}-a\right)^2+b^2}\right) - b - a\ln b\Bigg].
\end{aligned}$$

In order to compute the integrals of (C.4), we recall that

$$\int_0^b \frac{d\beta}{\sqrt{b^2-\beta^2}} = \sin^{-1}\frac{\beta}{b}\bigg|_0^b = \frac{\pi}{2}\,,$$

$$\int_0^b \frac{\beta d\beta}{\sqrt{b^2-\beta^2}} = -\sqrt{b^2-\beta^2}\bigg|_0^b = b\,,$$

$$\int_0^b \frac{\beta^2 d\beta}{\sqrt{b^2-\beta^2}} = -\frac{\beta}{2}\sqrt{b^2-\beta^2} + \frac{b^2}{2}\sin^{-1}\frac{\beta}{b}\bigg|_0^b = \frac{\pi b^2}{4}\,,$$

$$\begin{aligned}\int_0^b \frac{\ln(K^2+\beta^2)}{\sqrt{b^2-\beta^2}}\,d\beta &= \begin{bmatrix}\beta = bt\\ d\beta = bdt\end{bmatrix}\\ &= \int_0^1 \frac{\ln K^2 + \ln(1+\frac{b^2}{K^2}t^2)}{\sqrt{1-t^2}}\,dt\\ &= \pi\ln K + \int_0^1 \frac{\ln(1+\frac{b^2}{K^2}t^2)}{\sqrt{1-t^2}}\,dt\\ &= \pi\ln K + \pi\ln\frac{1+\sqrt{1+\frac{b^2}{K^2}}}{2}\\ &= \pi\ln\frac{K+\sqrt{K^2+b^2}}{2} \quad \text{(see [21], sec. 4.295.38)},\\ \int_0^b \frac{\beta\arctan K\beta}{\sqrt{b^2-\beta^2}}\,d\beta &= -\sqrt{b^2-\beta^2}\arctan K\beta\bigg|_0^b + K\int_0^b \frac{\sqrt{b^2-\beta^2}}{1+K^2\beta^2}\,d\beta\\ &= Kb^2\int_0^1 \frac{\sqrt{1-t^2}\,dt}{1+(K^2b^2)t^2}\\ &= \frac{\pi}{2}\left[\sqrt{b^2+\frac{1}{K^2}} - \frac{1}{K}\right].\end{aligned} \tag{C.6}$$

So, the computation of the integral for the first equation in (C.4) yields

$$\begin{aligned}&\frac{\pi}{4}\left[\ln\frac{a-T+\sqrt{(a-T)^2+b^2}}{2} + \ln\frac{a+T+\sqrt{(a+T)^2+b^2}}{2}\right]\\ &\quad -\frac{\pi}{2}\left[\ln\frac{\frac{\mu}{2}-a+\sqrt{(\frac{\mu}{2}-a)^2+b^2}}{2} + x + 4ta\right].\end{aligned} \tag{C.7}$$

Substituting this together with the first integral from (C.5) into (C.2) yields

$$\ln\left[a - T + \sqrt{(a-T)^2 + b^2}\,\right] + \ln\left[a + T + \sqrt{(a+T)^2 + b^2}\,\right] = 2[\ln b + x + 4ta]\,. \tag{C.8}$$

The computation of the integral for the second equation in (C.4) yields

$$\begin{aligned} &\frac{a\pi}{4}\Bigg(\ln\left[a - T + \sqrt{(a-T)^2 + b^2}\,\right] + \ln\left[a + T + \sqrt{(a+T)^2 + b^2}\,\right] \\ &\qquad - 2\ln\left[\frac{\mu}{2} - a + \sqrt{\left(\frac{\mu}{2} - a\right)^2 + b^2}\,\right]\Bigg) \\ &\qquad - \frac{\pi}{4}\left[\sqrt{(a-T)^2 + b^2} + \sqrt{(a+T)^2 + b^2} - 2\sqrt{\left(\frac{\mu}{2} - a\right)^2 + b^2} - \mu\right] \\ &\qquad + \frac{\pi}{2}[b - a(x + 4ta) + 2tb^2]\,. \end{aligned} \tag{C.9}$$

Substituting this together with the second integral from (C.5) into (C.2) yields

$$\begin{aligned} &a\ln\left[a - T + \sqrt{(a-T)^2 + b^2}\,\right] + a\ln\left[a + T + \sqrt{(a+T)^2 + b^2}\,\right] \\ &\qquad - \left[\sqrt{(a-T)^2 + b^2} + \sqrt{(a+T)^2 + b^2} - \mu\right] \\ &\qquad = 2a(x + 4ta) - 4tb^2\,. \end{aligned} \tag{C.10}$$

Combining (C.8) and (C.10), we get the system

$$\begin{cases} \sqrt{(a-T)^2 + b^2} + \sqrt{(a+T)^2 + b^2} = \mu + 4tb^2 \\ \left[a - T + \sqrt{(a-T)^2 + b^2}\,\right]\left[a + T + \sqrt{(a+T)^2 + b^2}\,\right] = b^2 e^{2(x+4ta)}\,. \end{cases} \tag{C.11}$$

In the particular case $\mu = 2$, this system becomes

$$\begin{cases} \sqrt{a^2 + b^2} = 1 + 2tb^2 \\ a + \sqrt{a^2 + b^2} = be^{(x+4ta)}. \end{cases} \tag{C.12}$$

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